Dynamic Hedging, Positive Feedback, and General Equilibrium

Dissertation zur Erlangung
des Grades eines Doktors der Wirtschaftswissenschaft
(Doctor rerum politicarum)
der Rechts- und Wirtschaftswissenschaftlichen Fakultät
der Universität des Saarlandes

vorgelegt von
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Saarbrücken 2001

Tag der Disputation: 27. Oktober 2000

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Vorwort

Ein umfangreiches Projekt wie das einer Promotion bedarf nicht nur eines guten Plans und des Ehrgeizes, diesen zu verfolgen. Vielmehr ist hierzu auch ein geeignets Umfeld erfordlich, in dem das Projekt gedeihen kann. Ein solches Umfeld kann man in der Regel jedoch nicht selbst schaffen. Dies verlangt nach einer Vielzahl von Helfern, die direkt oder indirekt, bewusst oder unbewusst, dafür Sorge tragen, dass das Projekt letztlich sein Ziel erreicht. In diesem Vorwort möchte ich all diesen Helfern danken, während die thematische Einführung dem Haupttext vorbehalten bleiben soll.

Danken möchte ich an erster Stelle meinem Doktorvater Herrn Prof. Dr. Jürgen Eichberger, dass er sich bereit erklärte, die Betreuung meiner Dissertation zu übernehmen. Er war es auch, der mich auf die äußerst interessante und spannende Thematik aufmerksam gemacht hat. Bedingt durch den Umstand, dass ich den größten Teil der Promotion fest im Berufsleben stand, gestaltete sich die Betreuung sicherlich nicht immer einfach. Nichtsdestotrotz hat er es stets verstanden, mir für meine Forschung die wesentlichen Impulse zu geben. Er ermöglichte es mir auch, dass ich in der Endphase der Anfertigung der Dissertation eine Assistentenstelle am Lehrstuhl für Wirtschaftstheorie der Universität des Saarlandes annehmen konnte. Dies hat die Fertigstellung der Dissertation nachhaltig beschleunigt.

Herrn Prof. Dr. Ralph Friedmann möchte ich danken, dass er sich bereit erklärte, neben Prof. Dr. Jürgen Eichberger die Begutachtung der Dissertation vorzunehmen. Nicht zuletzt auf Grund seiner kritischen Lektüre war es noch möglich, wesentliche Fehler vor Drucklegung der Dissertation zu beheben.

Herr Dr. Klaus Schindler stand mir regelmäßig als geduldiger und anregungsreicher Gesprächspartner zur Erörterung vielfältiger mathematischer Fragestellungen zur Verfügung. Er wurde hierbei nie müde, meine Aufmerksamkeit auf das Wesentliche zu lenken. Ihm möchte ich für seine fortwährende Unterstützung danken.

Elizabeth Freeland gebührt ebenfalls Dank, da sie nicht nur meine auf Englisch abgefassten Arbeitspapiere sondern auch die gesamte Dissertation hinsichtlich Grammatik und Schreibstil kritisch gelesen und mit großer Geduld und Sorgfalt verbessert hat. Auf diesem Wege habe ich ihr auch eine Erweiterung meiner eigenen englischen Sprachkenntnisse zu verdanken.

Die Herren PD Dr. Willy Spanjers, Dr. Jens Tapking, Dr. Christian Pfeil und Diplom-Volkswirt Daniel Quinten, alle ehemalige Assistenten des Lehrstuhls für Wirtschaftstheorie, standen mir im Laufe der Promotion häufig mit Rat und Tat bei vielen kleinen und größeren Problemen zur Seite. Die zahlreichen Diskussionen mit ihnen haben sicherlich die Dissertation an vielen Stellen verbessert. In diesem Zusammanhang möchte ich auch allen Teilnehmern des volkswirtschaftlichen Forschungskolloquiums an der Universität des Saarlandes danken, die sich im Rahmen meiner Vorträge zu verschiedenen Themen als kritische und kenntnisreiche Zuhörer erwiesen haben.

Meiner Familie und meiner Freundin möchte ich ganz besonders für ihre liebevolle Unterstützung über die Jahre hinweg danken. Ich weiß, dass meine Eltern, Iris und Adolf Hilpisch, meine Großeltern, Marlies und Johann Kaus, sowie meine Freundin Anke Alamanni stets darauf vertraut haben, dass ich dieses Promotionsprojekt erfolgreich abschließen werde. Das Wissen um ihr Vertrauen in mich hat mir sehr viel Kraft gegeben, ohne die mir die Promotion weitaus schwerer gefallen wäre. Meine Familie ebenso wie meine Freundin haben mich darüber hinaus in so vielfältiger Hinsicht unterstützt, dass man ihren Beitrag zum Gelingen dieses Projektes gar nicht überbewerten kann. Während der Anfertigung der Dissertation und insbesondere in der Endphase hat gerade meine Freundin großes Verständnis gezeigt, selbst dann, wenn gewisse Arbeiten wieder einmal viel mehr Zeit in Anspruch genommen haben als geplant. Sie erwies sich immer wieder als besonders ausdauernder und aufmerksamer Zuhörer.

Widmen möchte ich die Dissertation dem Andenken an meine Urgroßmutter Elisabeth Graf.

Part I

Introductory Part

Chapter 1

General introduction

Simply spoken, dynamic hedging is a device to manage risk as incurred, for example, by writing options. This thesis sheds light on dynamic hedging from different perspectives. One focal point of the thesis represents the study of dynamic hedging strategies in perfect and complete markets. It turns out that under a certain condition dynamic hedging strategies generate positive feedback. In this context, positive feedback means buying when prices rise and selling when they fall. Another focal point is the study of dynamic hedging in imperfectly liquid markets - or more precisely, in a general equilibrium framework - for the purpose of exploring the impact of dynamic hedging on financial markets.

We consider the analysis of dynamic hedging in a setting where markets are imperfectly liquid to be important for two main reasons. The first is that dynamic hedging when carried out on a large scale is likely to influence real financial markets. Therefore, it is worthwhile to gain an understanding of the economic implications of dynamic hedging. At first glance, it seems sensible to believe that dynamic hedging, because of its positive feedback property, contributes to excess volatility in financial markets. This is a striking point because market volatility is generally considered to be an appropriate measure for market stability. There is also evidence that dynamic hedging may even have contributed to recent financial crises like the stock market crash of 1987 or the crisis in 1998. Therefore, a deeper understanding of the economic implications of dynamic hedging is necessary to prevent such disruptions in the future and to address regulatory issues.

The second main reason is that the pricing and hedging formulas in which practitioners rely on in their everyday business are derived from models that are based on quite unrealistic assumptions about financial markets (e.g., perfect liquidity). The analysis of dynamic hedging in imperfectly liquid markets is a necessary prerequisite to assess the implications of relaxing the perfectly liquid markets assumption. Such an analysis may thereby help to improve existing pricing models and applied hedging procedures.

Before we survey recent work in this growing area of research, it seems helpful to address three important questions related to dynamic hedging.

Systematic behavior, such as positive feedback trading, is likely to arouse interest among economists seeking to explain individual behavior. A natural starting point for an economist who examines dynamic hedging might be to ask: *What are the underlying principles that make dynamic hedging strategies exhibit the positive feedback property*? This will also be our starting point in section 1.1.

As a matter of fact, positive feedback trading is inconsistent with the standard rationality hypothesis that behavior can be represented as the maximization of a suitably chosen utility function. Traditional approaches to explaining individual behavior (e.g., expected utility hypothesis) and market phenomena (e.g., efficient market hypothesis) obviously break down in the presence of dynamic hedgers or 'irrational' traders. This leads to the second question: *What should an economic model look like that is capable of incorporating irrational behavior like positive feedback trading*? An appropriate approach is portrayed in section 1.2.

When asked what impact positive feedback hedging has on financial markets, most economists would probably reply that it amplifies price movements. This intuition is clear. On the one hand, when a price rise is observed, dynamic hedgers cause an overshooting by making additional purchases. On the other hand, when the price of a security falls, dynamic hedgers contribute to an overshooting with additional selling. It is reasonable to assume that the mentioned effects crucially depend on the liquidity of the market in which dynamic hedging is implemented. Consequently, the third important question is: *What is the actual relationship between dynamic hedging and market liquidity*? Section 1.3 picks up this question.

Equipped with this background knowledge about dynamic hedging, section 1.4 sets out the plan for this thesis and sketches its structure. In section 1.4, we also argue why a general equilibrium model suggests itself as a natural place to analyze economic implications of dynamic hedging.

1.1 Replication and arbitrage

There is hardly any other breakthrough in financial economics that can compare with the one of Black and Scholes (1973) and Merton (1973) in terms of both approach and applicability. Fischer Black and Myron Scholes, in collaboration with Robert Merton, were the first to derive a closed form solution for the price of a European call option that is free of any ad-hoc elements. Their solution expresses a direct relationship between the price of a European call option and the relevant market parameters that influence it, such as the price of the underlying, the risk-less interest rate and so forth.[1] Before their pricing formula had become publicly available, option pricing was a rather mysterious task always involving ad-hoc elements. Up to now, a separate branch of the financial services industry has emerged that mainly relies on the NOBEL-prize winning BLACK / SCHOLES / MERTON theory. Yet the big success of their theory cannot be justified in terms of the pricing formula alone, but rather in terms of the *replication principle* and the *arbitrage argument* they applied to derive the formula.

After having outlined several common assumptions of standard contingent claim[2] pricing models, we want to discuss in this section the replication principle and the arbitrage argument that together play such a central role in deriving fair prices for contingent claims. Moreover, we want to give a brief overview of how these basic concepts may be applied in areas other than pricing.

As primitives, contingent claim pricing models generally entail a fixed set of available securities.[3] Agents regularly not explicitly modelled, can trade in these securities to transfer wealth through time. Within these models, a natural question is: What is a fair price for a given contingent claim? It is clear that the answer to this question crucially depends on the imposed model assumptions. In general, seven common assumptions can be identified that contingent claim pricing models share with respect to financial markets and agents interacting in these markets.[4]

1. Markets

    (a) *Perfect markets*: Markets are perfect in the sense that there are no transaction costs, that unlimited short-selling is possible and that securities are available in any fraction. In such markets, everything happens at light speed as well.

    (b) *Perfectly liquid markets*: Markets are perfectly liquid in the sense

[1] We reproduce their famous result in sub-section 5.3.2.

[2] We subsume, under the expression *contingent claim,* arbitrary claims contingent on future events. Examples are derivative securities whose prices depend on uncertain future prices of other securities. Another one is the uncertain amount of money that someone receives at some future date.

[3] HULL (1997) is a standard textbook on contingent claim pricing theory while the book of BRIYS, BELLALAH, MAI, and DE VARENNE (1998) contains some more recent research.

[4] We will sometimes refer to these assumptions as the 'standard assumptions'.

that buy or sell orders of arbitrary magnitude cannot affect security prices.

(c) *Complete markets*: Every contingent claim is attainable via trading in the available securities.

(d) *Arbitrage-free markets*: Trading in securities cannot produce something out of nothing, i.e., risk-less profits are impossible.

2. Market participants

(a) *Perfect competition*: All market participants act as price takers, or equivalently, all market participants are atomistic.

(b) *Symmetric information:* All market participants own the same information.

(c) *Complete information*: Market participants have complete information regarding relevant market parameters.

It is obvious that these assumptions do not draw a realistic picture of the real world. However, the advantage of working in such an idealized world is that very strong results are obtained. For example, Black and Scholes (1973) are able to show that in such a world the price of a European call option depends only on observable market parameters and does not depend on, let us say, the preferences or beliefs of market participants as one would probably expect.

For the sake of expositional ease, we want to explain the replication principle and the arbitrage argument in a very simple fashion rather than reproduce the original argument of Black and Scholes (1973).[5] To begin with, consider an economy with only two relevant dates, say today and tomorrow, in which the above seven assumptions are satisfied. Complete markets imply that a given option (or any other contingent claim) with maturity tomorrow can today be replicated by an appropriate combination of the available securities, a so-called replication portfolio. In other words, today there exists at least one combination of the available securities that has a payoff tomorrow identical to the payoff of the option. Because of complete and symmetric information, everyone in the economy can easily identify such a portfolio and compute the costs to set it up.

[5]In what follows, we will have in mind something called *two state option pricing.* This access to the reasoning behind replication and arbitrage was originally proposed by Sharpe (1978). We provide a numerical example for two state option pricing in subsection 4.4.1.

At this point, the absence of arbitrage comes into play. It forces the price of the option to equal the price of the replication portfolio. To see this, consider the two other possibilities. Assume, for instance, that the price of the option is higher than the price of the replication portfolio. Selling the option short and buying the replication portfolio would lock in a risk-less profit as high as the difference between the two prices. The profit is actually risk-less because the payoff of the option tomorrow and the payoff of the replication portfolio tomorrow will compensate each other perfectly. Conversely, assume that the option is cheaper than its replication portfolio. Selling the replication portfolio short and buying the option would then lock in a risk-less profit as high as the price difference. In this case, the reason for the profit being risk-less is the same as before. Non-satiated investors would continually try to pursue such risk-less arbitrage strategies in order to achieve infinite wealth. For markets to be in equilibrium, this must be excluded. Thus, the only price for the option consistent with the absence of arbitrage and market equilibrium is the price of the replication portfolio.[6]

Interestingly, this line of argument proves robust in much more general settings. For example, if there are more than two dates and markets are still complete, then there exists a dynamic replication strategy. That is to say a sequence of replication portfolios generating a cash-flow identical to the cash-flow of the given option. As we see, the static argument applied in the two date case can be repeated as many times as necessary. The absence of arbitrage ensures then that at any date, the price of the option equals the price of the replication portfolio at that date. Cox, Ross, and Rubinstein (1979) propose this route to option prices. In terms of mathematical sophistication, their derivation of a pricing formula for European call options is much simpler than that of Black and Scholes (1973).

Another major breakthrough with respect to contingent claim pricing is the observation that pricing formulas - as obtained by Black and Scholes (1973) or Cox, Ross, and Rubinstein (1979) - allow for probabilistic interpretations. To be more precise, the pricing formulas are expectations of the respective contingent claim's payoff under an appropriately chosen probability measure. The defining property of this so-called *risk-neutral probability measure* or *martingale measure* is that risky securities, such as stocks, have an expected return that equals the risk-less interest rate. In other words, the discounted price processes are martingales under this probability measure.

[6] We should note that the original argument of Black and Scholes (1973) differs a bit from the one outlined here. They argue that in complete markets a given European call option can be combined with the underlying security in such a way that the overall position becomes risk-less. In the absence of arbitrage, the overall position must therefore yield the risk-less interest rate.

Path-breaking work in this regard is due to HARRISON and KREPS (1979).[7] DUFFIE (1996, xiv) notes that "... Harrison and Kreps (1979) gave an almost definitive conceptual structure to the whole theory of dynamic security prices."

Although the assumptions under which the replication principle and the arbitrage argument perfectly apply are rather restrictive, standard models relying on these assumptions are the most popular models in practice. A preferred approach of practitioners for overcoming their inherent shortcomings is to adjust the input parameters heuristically instead of improving the models themselves. Nevertheless, a tremendous amount of research has been conducted that deals with diverse generalizations of contingent claim pricing models. The relaxation of one or more of the above listed assumptions is central to this line of literature.[8]

We now turn to other possible areas of application for the replication principle and the arbitrage argument. We briefly want to sketch out three areas: *dynamic hedging*, *synthesizing* and *information discovery,* the first of which is of particular importance for our purposes.

The pricing of contingent claims is still an important issue for practitioners in the financial services industry. However, another area of application, *dynamic hedging*, has become at least equally important. Dynamic hedging, as a device for buyers and sellers of contingent claims, to manage risk has become a financial industry-wide practice. For example, consider a financial institution that has sold European call options to its customers for which no liquid market exists. The financial institution faces the risk of a rising underlying price since it may incur losses if the price rises high enough. The institution can dynamically hedge the sold options to insure against the losses. In order to achieve such a hedge, it has to implement a trading strategy, involving the underlying security and a risk-less bond, whose cash-flow perfectly matches the cash-flow of a long position in the options. Provided

[7] Noteworthy extensions can be found in HARRISON and PLISKA (1981) as well as COX and HUANG (1989), among others. For comprehensive treatments of the martingale approach, refer to PLISKA (1997) or KARATZAS and SHREVE (1998). We will introduce this approach to finance in part II of the thesis.

[8] Pioneering work on the incorporation of transaction costs into contingent claim pricing models is found in LELAND (1985). In chapter 3, we review several articles about dynamic hedging in imperfectly liquid markets. MAGILL and QUINZII (1996) provide a survey of recent developments in the theory of incomplete financial markets. FREY (1997) surveys the large body of literature about stochastic volatility.

Recently, a lot of effort has been put into the development of a unifying framework capable of incorporating different market frictions simultaneously (e.g., transactions costs, short sale restrictions, incomplete markets). CVITANIC (1997) reviews this framework which is found in the literature under the expression 'constrained markets'.

markets are complete and frictionless enough, the trading strategy generates at maturity a long position in the options. Combined with the short position of the sold options, the financial institution has a risk-less net position.[9] This illustrates why sound hedging techniques have been the enabler for the fast growing derivatives industry in both the exchange segment and the over-the-counter (OTC) segment.[10] In regard to the OTC segment, DUFFIE (1998, 419) remarks:

> "Investment banks routinely sell securities with embedded options of essentially any variety requested by their customers, and then cover the combined risk associated with their net position by adopting dynamic hedging strategies."

A closely related area of application is *synthesizing of contingent claims.* Suppose a financial institution identifies a need for a derivative security that is not publicly available for sale. By implementing a dynamic hedging strategy, as described above, it can generate the needed derivative by itself. In such a case, i.e., where there is no real counterpart to the dynamic hedging strategy, the strategy is said to synthesize the desired derivative. The most popular form of such a trading strategy is portfolio insurance. Portfolio insurance synthesizes a European put option to ensure that the value of a given portfolio of securities does not fall under a certain floor. We will return to portfolio insurance later in the chapter. In this thesis, however, we subsume both types of trading strategies, dynamic hedging and synthesizing, under the expression *dynamic hedging* since there is apparently no formal difference between the two.

Yet another area of application is *information discovery.* Market participants may interchange the role of variables in pricing models. If, for example, someone is interested in the volatility of a security, he / she can use available option prices as input into a pricing formula to determine the volatility of the underlying security according to that formula. One would then speak of the *implied volatility.*[11]

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[9] However, one should not forget that *model risk*, i.e., the risk of applying an inadequate dynamic hedging scheme, arises as a new source of risk. GREEN and FIGLEWSKI (1999), for example, study this kind of risk.

[10] SCHOLES (1998, 350) describes the two different segments as follows: "Financial institutions in the OTC industry offer customized derivative products to meet the specific needs of each of their clients; the exchange industry offers standardized products to reach a richer cross section of demand."

Interestingly, the foundation of the Chicago Board Options Exchange (CBOE) and the publication of the seminal article of BLACK and SCHOLES (1973) roughly coincided.

[11] RUBINSTEIN (1994), for instance, is a recent study exploring this area of pricing model application.

1.2 Informed and uninformed trading

Ever since its formulation, the theory of efficient markets has spurred a lot of controversy among theoretical and empirical researchers, as well as among practitioners. In the following, we will take a glance at this theory and some of the reasons for the continuing controversy. Moreover, we will briefly characterize the noise trader approach to finance. This approach seems capable of resolving some puzzles that surround the theory of efficient markets. As we will argue, several empirical findings and market phenomena, like excess volatility and dynamic hedging, are compatible with this more recent approach, whereas they are incompatible with the still dominating efficient markets paradigm.

In principle, the theory of efficient markets states that asset prices in financial markets fully reflect information available to the investment community. One generally distinguishes three forms of the *efficient market hypothesis* (EMH). The *weak* form says that security prices reflect all information contained in public market data, such as past prices and trading volumes. The *semi-strong* form postulates that prices reflect all publicly known pieces of information, such as market data as well as earnings, dividends, etc. Finally, the *strong* form says that security prices reflect all available information, either public or private.[12]

Of course, the theory of market efficiency, provided it proves robust, has far-reaching implications for almost all players in financial markets. A rational trader, for instance, would never try to beat the market performance without having fundamentally new information. Trading triggered by news about fundamentals is usually referred to as *informed* trading. To avoid misconceptions: the EMH does not require that a priori all traders be informed. Markets themselves act as a transmission mechanism for new information.[13]

Because of its important implications, it is no wonder that a lot of empirical studies have been undertaken to test the different forms of the EMH. The early theoretical and empirical work is surveyed in FAMA (1970). However, the more recent work is surveyed in its successor article FAMA (1991). Empirical evidence regarding the robustness of the EMH is mixed. It delivers arguments for both advocates of the theory and opponents. As a result, there is an ongoing debate about the interpretation of the empirical results

[12]In chapter 10, JONES (1998) introduces market efficiency and related fields.

[13]Regarding this mechanism, GROSSMAN and STIGLITZ (1980, 393) note: "When informed individuals observe information that the return to a security is going to be high, they bid its price up, and conversely when they observe information that the return is going to be low. Thus the price system makes publicly available the information obtained by informed individuals to the uninformed."

and about the appropriateness of the applied methods to test the EMH. Some, like GROSSMAN and STIGLITZ (1980), even argue that informationally efficient markets are impossible.

Much of the controversy is caused by conceptual problems in testing the EMH. The most challenging issue in testing the EMH is that it cannot be tested as a stand-alone; an equilibrium model must be utilized to decide upon 'fair' prices.[14] As a matter of fact, this joint-hypothesis problem makes it quite impossible to decide whether an anomaly, in terms of the respective equilibrium model, classifies as a supporting argument or a rejecting one in regard to the EMH. Indeed, the number of anomalies reported in the empirical literature is impressive and some anomalies still await plausible explanations.[15] Therefore, it is quite natural that researchers have tried, and continue to try, to develop alternative approaches to market equilibrium. The hope is that alternative approaches are capable of explaining alleged inefficiencies and anomalies.

By now, a very popular alternative to the standard rationality hypothesis, which builds the basis for common equilibrium models, is the *noise trader approach* as presented in BLACK (1986) or SHLEIFER and SUMMERS (1990), among others.[16] Contrary to the standard rationality hypothesis, the noise trader approach explicitly allows for a fraction of the investment community to be irrational and to conduct *uninformed* trades, i.e., trades based on noise. In this context, noise can be anything but fundamental news that triggers actions of traders.

According to the EMH, rational investors can exploit mispricings caused by irrational investors. Consider, for example, a stock that trades at a fundamentally justified price, i.e., its intrinsic value.[17] Irrational traders, forming beliefs on noise, may bid down its price by accumulated sell orders. According to the argument, if the stock price falls far enough, then rational traders step in to buy the stock. After the price has returned to the rational level,

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[14]FAMA (1991, 1576) remarks in this respect: "This point ... says that we can only test whether information is properly reflected in prices in the context of a pricing model that defines the meaning of 'properly.' As a result, when we find anomalous evidence on the behavior of returns, the way it should be split between market inefficiency or a bad model of market equilibrium is ambiguous."

Refer to CAMPBELL, LO, and MACKINLAY (1997) or CUTHBERTSON (1996) for details on the theoretical background of EMH tests.

[15]BODIE, KANE, and MARCUS (1996) discuss in section 12.4. several anomalies, such as the 'January effect' or the 'small firm effect'. For further details on this topic, consult FAMA (1991) and the many references given therein.

[16]KYLE (1985) introduces the expression *noise trader.*

[17]Refer to chapter 4 of CUTHBERTSON (1996) for a description of alternative methods to determine the intrinsic or fair value of a security.

this creates a risk-less arbitrage profit as high as the difference between the actual price and the fundamentally justified price, i.e. the rational price. Unless the arbitrage opportunity has disappeared, rational traders will keep on buying. Common models of equilibrium (implicitly) assume that this demand is perfectly elastic so that prices are always on their fundamentally justified level.

In contrast, the noise trader approach rests on the central assumption that arbitrage is limited due to two different kinds of risk, as SHLEIFER and SUMMERS (1990, 21) explain. The first is *fundamental risk*, meaning that changes in fundamentals may suddenly change the intrinsic value of a security. The second is *noise trader risk*, as investigated in DELONG, SHLEIFER, SUMMER, and WALDMANN (1990a). It expresses the possibility that uninformed traders may bid the price even further away from the intrinsic value. As a consequence of these types of risk, demand by rational traders is imperfectly elastic, thereby limiting arbitrage. Uninformed trading - contradicting the EMH - may therefore have a persisting impact on security prices. Empirical evidence suggests that this approach is consistent with market realities.[18]

A very common form of uninformed trading in the financial marketplace is *positive feedback* trading. A trader following a positive feedback trading strategy buys securities after a price rise and sells securities after a price fall. Such trades are executed contingent on observed price movements rather than on fundamental news. DELONG, SHLEIFER, SUMMER, and WALDMANN (1990b, 381-382) list four different types of strategies that lead to positive feedback:

> "These strategies include portfolio choice based on extrapolative expectations, the use of stop-loss orders, purchases on margin which are liquidated when the stock price drops below a certain point, as well as dynamic trading strategies such as portfolio insurance."

The last one on this list of four is of course the most important one considering the topic of this thesis. In chapter 5, we will demonstrate that not only portfolio insurance induces positive feedback trading, but that this holds true for dynamic hedging of arbitrary contingent claims with convex payoffs. It is intuitively plausible that positive feedback trading, if simultaneously implemented by enough agents, can disturb markets in a systematic way, i.e. prices tend to overshoot. The argument supporting this statement is as follows. On one hand, after a price rise, positive feedback traders become active and buy securities. This additional demand causes prices to rise even further, thus

[18] SHLEIFER and SUMMERS (1990, 22-23) cite some empirical findings in this respect.

they overshoot. On the other hand, after a market decline, positive feedback traders appear and sell securities, which leads to a further decline. Again, prices overshoot. DeLong, Shleifer, Summer, and Waldmann (1990b) argue that the effect of positive feedback trading may even be amplified by rational traders front-running the feedback trades to profit from anticipated price overreactions.

Overreactions of stock market prices have been reported by DeBondt and Thaler (1985) and DeBondt and Thaler (1987), among others. Statistically, overreaction means that security returns show positive serial correlation over some short period of time. However, there is also empirical evidence of negative serial correlation for longer periods. If it turns out that short periods of positive serial correlation and long periods of negative serial correlation coexist, then security returns tend to be mean-reverting. Indeed, researchers have found considerable evidence for mean-reversion in security returns.[19]

Moreover, the presence of positive feedback traders in markets could imply higher market volatilities since positive feedback causes overreactions in both directions. DeLong, Shleifer, Summer, and Waldmann (1989) argue that positive feedback trading may therefore be a possible explanation for excessive volatility as reported by Shiller (1981) and others.[20] Yet Porterba and Summers (1986) find that volatility shocks vanish rapidly. This observation is in line with the mean-reversion hypothesis. All in all, we can dare to assert that the impact of positive feedback trading is of temporary nature at most. When analyzing dynamic hedging in imperfectly liquid markets, it seems adequate enough to take into consideration the temporary nature of effects from dynamic hedging.

In part III of the thesis, where we analyze dynamic hedging in a general equilibrium context, the empirical findings as mentioned before, are indeed accounted for. This is due to the requirement that security price processes be tied to fundamentals at the terminal date of the model economy. Before the terminal date, however, prices may be influenced by uninformed trading. Considering this, the equilibrium models that are investigated in part III are principally consistent with both overreaction and mean-reversion.

[19] The article by DeBondt and Thaler (1989) gives a brief survey of this line of empirical literature.

[20] A topic related to excessive volatility is that of speculative bubbles. Flood and Hodrick (1990) outline several aspects of this relationship.

1.3 Dynamic hedging and market liquidity

In the last section, we have argued that positive feedback trading strategies, like dynamic hedging, may cause overreactions in security prices and thereby potentially increase market volatilities. It is reasonable to assume that the likelihood of the realization for such volatility increases as well as its eventual magnitude are basically determined by two factors. The first factor is the *liquidity of markets* in which positive feedback traders are active. The second is the *market weight of traders* following positive feedback trading strategies. In this section, we will discuss the importance of these two factors for dynamic hedging from a theoretical (sub-section 1.3.1) and an applied perspective (sub-section 1.3.2). The emphasis in sub-section 1.3.1 lies on the liquidity aspect while it focuses more on the market weight aspect in sub-section 1.3.2.

Throughout this section we use the word *liquidity* in the following sense. We say that a security market is *perfectly liquid* if buy or sell orders cannot affect security prices. Accordingly, markets are said to be *imperfectly liquid* if security trading can affect prices. Similarly, a single market is said to be *more liquid* in situation 1 compared to situation 2 if equal buy or sell orders move security prices less in situation 1 than in situation 2. It will also prove useful to associate market liquidity with the elasticity of aggregate demand in that market.

1.3.1 A theoretical perspective

In this sub-section, we examine a rather stylized model of a financial market. In the benchmark case, all seven assumptions discussed in section 1.1 remain in force. To highlight the importance of market liquidity for dynamic hedging and pricing of contingent claims we then drop the assumption of perfectly liquid markets.[21] This allows us to provide most of the intuition behind the results derived in part III of the thesis. To keep our argument as simple as possible, we restrict the analysis to a static (or one shot) economy. However, the argument carries over to a dynamic setting in a straightforward manner.

In a financial market characterized by the seven assumptions as found on page 13, two different groups of agents interact with each other. Agents trade today in a stock and a risk-less bond. The stock is in fixed supply of $\overline{a}$ shares and pays tomorrow an uncertain liquidating dividend while the bond is in zero net supply. One group of the agents forms rational expectations about the uncertain future returns of the stock. Agents of this group are called *non-hedgers*. The other group consists of agents who hedge a given

[21] Pritsker (1997) examines the relationship between positive feedback trading and market liquidity as well.

European call option on the available stock by investing in an appropriate replication portfolio. Since markets are complete, it is guaranteed that the option can be replicated. Agents out of this group are called *hedgers*.

We denote by ϕ^M the non-hedgers' aggregate stock demand and by ϕ^H the hedgers' aggregate stock demand.[22] Clearly, the stock market is in equilibrium if aggregate supply equals aggregate demand,

$$\phi^M + \phi^H = \overline{a}. \tag{1.1}$$

It is now important to realize that in complete markets the stock demand of the hedgers is independent of the present stock price. It only depends on the uncertain liquidating dividend that is paid tomorrow and, of course, on tomorrow's payoff of the call option.[23] The present stock price does not come into play until the hedgers want to evaluate the replication portfolio to deduce the price of the option. Conversely, it is reasonable to assume that the non-hedgers' demand depends on the stock price today. If we combine these considerations with (1.1) we finally end up with,

$$\phi^M(S_0) = \overline{a} - \phi^H, \tag{1.2}$$

where S_0 denotes the stock price today. (1.2) states that S_0 must be in equilibrium such that the stock demand by the non-hedgers ϕ^M equals aggregate supply of the stock $\overline{a}$ adjusted for the stock demand by the hedgers ϕ^H. Since the hedgers hedge European *call* options, their stock demand must be strictly positive, $\phi^H > 0$.[24]

As the benchmark case for our analysis we have chosen the case of perfectly liquid markets as already mentioned. In the present context, the assumption of perfectly liquid markets is tantamount to the assumption of perfectly elastic stock demand by the non-hedgers.[25] In other words, at a given price S_0^*, non-hedgers are willing to buy and sell the stock in any quantity. Figure 1.1 displays such a situation. Consider first the case where there is no demand by hedgers. Equilibrium in the stock market is then determined by the intersection of the non-hedgers stock demand with the aggregate supply (point A). If there is strictly positive hedge demand of ϕ^H, then equilibrium is obtained in point B and for a double as high hedge demand, $2 \cdot \phi^H$, in point

[22] We abstract here from possible problems in aggregating among agents.

[23] At this stage, the statement is not more than mere assertion but chapter 5 verifies this assertion for the Cox, Ross, and Rubinstein (1979) model.

[24] See, for example, section 14.5 of Hull (1997).

[25] This interpretation of market liquidity is also found in Gammill and Marsh (1988, 42), Genotte and Leland (1990) or Frey (1995, 1-2) to name just a few.

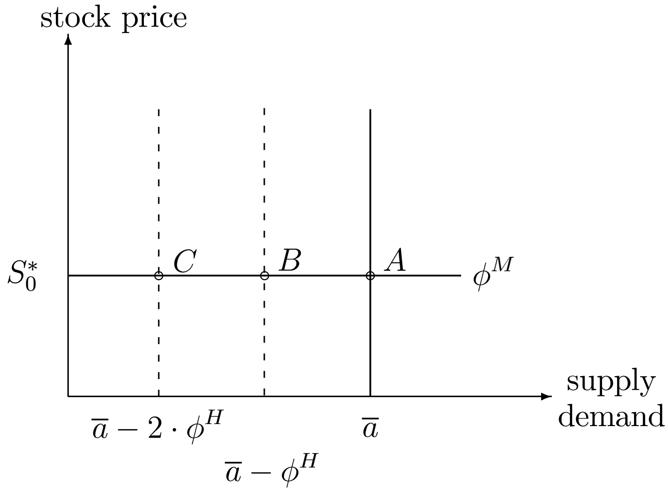

Figure 1.1: Dynamic hedging in a perfectly liquid market.

C. Notably, the figure illustrates that in perfectly liquid markets hedge demand cannot affect the equilibrium stock price. No matter at what point the stock market equilibrium prevails, the equilibrium stock price remains at S_0^*.

Consider now the more realistic case of an imperfectly liquid stock market, or equivalently, of imperfectly elastic stock demand by the non-hedgers. The bond market, however, should remain perfectly liquid. For simplicity, assume a strictly downward-sloping stock demand function. The picture that emerges in such a market is provided in figure 1.2. As before, consider first the case with no hedging at all. Equilibrium in this case is indicated through point A' where stock demand by the non-hedgers equals aggregate stock supply. If we introduce strictly positive hedge demand of ϕ^H, the equilibrium moves to point B', and if we even double the hedge demand to $2 \cdot \phi^H$, equilibrium moves further to point C'. The corresponding prices equating supply and demand are S_0^*, S_0' and S_0'', respectively. As we see in figure 1.2, strictly positive hedge demand by the hedgers causes the equilibrium stock price to increase from S_0^* to S_0'. It also increases with increasing stock demand by the hedgers (from S_0' to S_0''). We observe that dynamic hedging can move prices in imperfectly liquid markets when applied on a large scale. This contradicts the paradigm of standard contingent claim pricing models where such feedback effects are de facto ruled out by the assumption of perfectly liquid markets.

Imperfectly liquid markets also have important implications for the pricing of derivative securities. To be precise, assume that ϕ^H now represents the number of shares of the stock needed to hedge exactly one European call

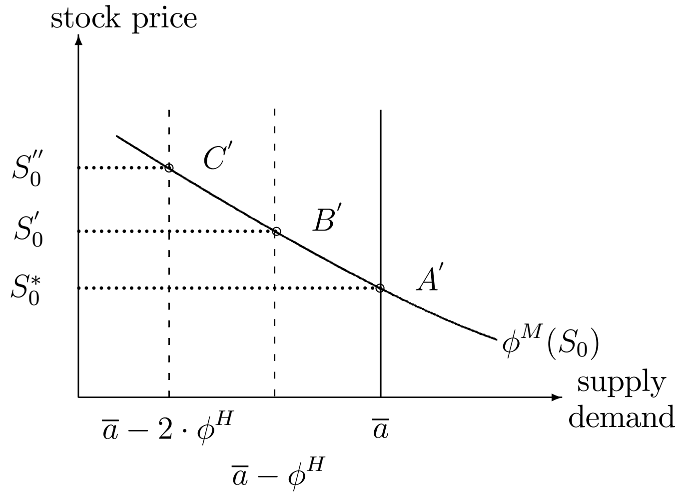

Figure 1.2: Dynamic hedging in an imperfectly liquid market.

option. Furthermore, let $\widetilde{\phi}^H$ represent the number of bonds needed for that purpose. By the replication principle and the arbitrage argument of section 1.1, a hedger can derive the price $C_0^{'}$ of a single call option easily from,

$$C_0^{'} = S_0^{'} \cdot \phi^H + \widetilde{S}_0^{*} \cdot \widetilde{\phi}^H. \tag{1.3}$$

$\widetilde{S}_0^{*}$ denotes the price of a unit of the bond today. Since the bond market is perfectly liquid by assumption, this price is not influenced by the bond demand of the hedgers. The arbitrage argument implies that if (1.3) is violated, risk-less arbitrage profits are possible. Since this should be impossible in equilibrium, (1.3) gives the price of one call option contract. If one uses this argument to derive the price $C_0^{''}$ of two option contracts, one similarly obtains,

$$C_0^{''} = S_0^{''} \cdot (2 \cdot \phi^H) + \widetilde{S}_0^{*} \cdot (2 \cdot \widetilde{\phi}^H). \tag{1.4}$$

Here we bear in mind that the stock and bond demand of hedgers is independent of the respective prices today. We can therefore simply double the bond and stock positions. According to (1.4), the price of one option contract consistent with no arbitrage is $\frac{1}{2} \cdot C_0^{''}$. Unfortunately, this price does not equal the price of the option contract if only one single option is replicated. The option price in the case where two options are hedged exceeds the option price in the case where only one option is hedged. Formally,

$$\frac{\frac{1}{2} \cdot C_0^{''}}{C_0^{'}} = \frac{S_0^{''} \cdot \phi^H + \widetilde{S}_0^{*} \cdot \widetilde{\phi}^H}{S_0^{'} \cdot \phi^H + \widetilde{S}_0^{*} \cdot \widetilde{\phi}^H} > 1,$$

where we use $S_0'' > S_0'$ (see figure 1.2).

In summary, the appealing Black / Scholes / Merton approach obviously fails in imperfectly liquid markets. It no longer delivers a unique price for a given contingent claim even though markets are complete and replication is possible. In imperfectly liquid markets, the price of a contingent claim also depends on the aggregate demand for the claim - a result we formally derive in chapter 6. Moreover, dynamic hedging of contingent claims influences markets considerably if one drops the assumption of perfect liquidity. In our rather simple setting, rises in the stock demand by the hedgers make stock prices rise as well.

We now turn to empirical and anecdotal evidence supporting our claims made in this sub-section.

1.3.2 An applied perspective

There is no doubt that positive feedback trading and, in particular, dynamic hedging takes place in financial markets. There should also be no doubt that markets are imperfectly liquid in general. One has to be a bit more careful, however, in asserting that dynamic hedging is likely to disturb asset prices in financial markets and that the Black / Scholes / Merton approach is not applicable to real markets. A question that has to be addressed first is whether real market conditions can prevail, such that dynamic hedging can affect prices in financial markets. In this sub-section we therefore provide evidence that the potential of dynamic hedging to impose feedback on markets can indeed be realized. Due to little empirical work on this topic, we additionally take a brief look at two events that have attracted considerable attention in both the academic and the non-academic world. The first one is the stock market crash of October 19, 1987 and the second one is the near-collapse of Long-Term Capital Management (LTCM) in 1998.

Of course, we can not paint a full picture of all facets of dynamic hedging and liquidity related problems here and we do not even intend to do so. We can, however, highlight those details that may help in gaining an understanding of how dynamic hedging and market liquidity are related in reality. As a supplementary text for the issues raised in this sub-section, the recently published book by Jacobs (1999) may be consulted. It is presently one of the most comprehensive informal treatments of dynamic hedging and its economic implications. The long list of references given in the book, including theoretical, empirical and applied material, is noteworthy as well.

Empirical evidence

Among empirical researchers, the impact of dynamic hedging on financial markets has experienced only little attention. KAMBHU (1998) is the first, and to our knowledge, the only one to empirically assess feedback effects from dynamic hedging.[26] He focuses on the interest rate options markets. The reasoning he gives for this choice is that "[t]he concentration of sold options among dealers ... makes it an ideal place to explore how dealers' hedging of options affects underlying markets." KAMBHU (1998, 36).

Even though KAMBHU (1998, 36) finds that, in general, market liquidity is sufficient to absorb the hedge demand by the option dealers, he finds contrary evidence for the mid-term segment of the yield curve. He concludes:

> "At maturities beyond three years, however, if dealers fully rebalance their hedge positions, dynamic hedging in response to a large interest rate shock could be of significant volume relative to transaction volume and outstanding contracts in the most liquid trading instruments. At this segment of the yield curve, the potential for positive feedback when a large interest rate shock occurs cannot be dismissed." KAMBHU (1998, 46).

Comparing option prices, he also discovers that the relatively low liquidity in the mid-term segment seems to be priced in by the dealers. KAMBHU (1998, 44) reports that in this segment, demand generated by dynamic hedging can amount to 21% of the overall daily trading volume if an interest rate shock of 75 basis points occurs. However, he notes on page 46 that "[t]he ultimate impact of dealers' dynamic hedging would depend on the relative size of different types of market participants." Although not exhaustive, the results of KAMBHU (1998) clearly support several findings of recent theoretical studies of dynamic hedging.[27]

To finally judge the results of KAMBHU (1998), one has to take into account that interest rate markets are generally among the most liquid markets. The observed effects would have probably been much stronger if the analysis was carried out in a less liquid market, for instance the market for a single stock.

The practitioners' view

Standard option pricing models, such as those proposed in BLACK and SCHOLES (1973) or COX, ROSS, and RUBINSTEIN (1979), do not give hints

[26] KAMBHU (1998, 37) himself remarks that " ... no empirical proof exists that positive feedback affects market prices ..."

[27] See chapter 2.

regarding the importance of liquidity for dynamic hedging and option pricing. Empirical evidence is rare as well. Nevertheless, practitioners relying on standard models in their everyday work are aware of the importance of market liquidity. The following quote illustrates this very well.

> "Von entscheidender Bedeutung ist zweifelsohne die Liquidität des Basiswertes. Sie spielt auch schon im Vorfeld - also bei der Konstruktion des neuen Scheins - eine wichtige Rolle ... Bei einem einfachen Call Warrant auf die DaimlerChrysler-Aktie können wir mit einem Auftrag schon 10000 Aktien zu einem einheitlichen Preis über Xetra ordern. Bei anderen Titeln sind es pro Auftrag und Kurs manchmal nur 1000 bis 2000 und weniger. ... Der Spread spiegelt den Grad der Liquidität des Basiswertes wider. Je illiquider die zugrundeliegende Aktie zum Beispiel ist, desto größer sind einerseits die Geld- / Briefspannen für die betreffende Aktie und anderseits die Spreads für den jeweiligen Schein ..." HANDELSBLATT (October 28, 1999, 48).

These statements, made by derivatives executives of investment bank WARBURG DILLON READ, emphasize that there may exist considerable differences in the liquidity of common stocks. As the executives point out, differences in the liquidity of underlying markets make it necessary to adjust option prices accordingly. Our abstract discussion in the previous sub-section argues why this is reasonable. Similarly, KAMBHU (1998, 41) discovers that "[f]or dynamic hedge adjustments, dealers are likely to use the most liquid instruments as hedging vehicles."

This shows that market participants are actually aware of the importance of market liquidity for the hedging and pricing of derivative instruments. Yet financial risk management in banks and corporations usually focuses on market risk management and credit risk management.[28] Obviously, liquidity risk seems less important.[29] Nevertheless, central banks consider this kind of risk very important. In the final report of a joint research project about market liquidity, positive feedback trading and market volatility, central bankers involved in the project describe their motivation for the project as follows:

> "A desire to better understand the dynamics of market liquidity motivated the research project. If a growing number of market

[28] *Market risk* refers to the risk that security prices, interest rates, etc. can change in an unfavorable way, whereas *credit risk* means the risk that a counterparty may fail to meet its obligation.

[29] DALVI and MASSARO (1999, 49) define *liquidity risk* as " ... the risk that a firm or party will be unable to meet its cash-flow when it is due, largely because it cannot receive full value for financial instruments that it is forced to sell."

participants are relying on market liquidity in their investment and risk management choices, then the robustness of that market liquidity would appear to be an important issue. One set of market participants who rely on market liquidity are those firms engaged in dynamic trading strategies, such as dynamic hedging or portfolio insurance. Previous research has highlighted the possibility that such strategies could, at times, have adverse repercussions for market functioning." BANK FOR INTERNATIONAL SETTLEMENTS (1997, 1).

The stock market crash of 1987

In the introduction of his survey article about stock market crashes, KLEIDON (1995, 465) writes:

> "Stock market crashes, defined as precipitous declines in value for securities that represent a large proportion of wealth ..., are rare, difficult to explain, and potentially catastrophic. During four trading days in the crash of October 1987, the U.S. stock market fell by about thirty percent, wiping out roughly one trillion dollars of equity. On October 19 alone, Black Monday, the market fell by over twenty percent."

A market decline as sharp as the crash of October 1987 is likely to draw a lot of attention; particularly since the crash of October 1987 was "... the largest one-day drop in the history of major stock market indexes from February 1885 through the end of 1988." [SCHWERT (1990, 77)] Indeed, the 1987 stock market crash has initiated a lot of studies carried out by governmental authorities, financial market players and academic researchers. Some of the theoretical work, explicitly aimed at explaining the crash, will be reviewed in chapter 2, section 2.4.

Even though the effort to explain the stock market crash has been tremendous and still continues to be so, things are far from being completely clear. Without question, however, dynamic hedging played an important role during the crash. A very popular dynamic hedging program at that time was portfolio insurance, a dynamic hedging strategy that, as already mentioned, replicates a European put option. These strategies were used to insure security portfolios against a fall in value under a predetermined floor. At first sight, their appeal to investors is understandable.[30] In theory, they are

[30] LELAND (1980) as well as BENNINGA and BLUME (1985) study the question for whom the implementation of portfolio insurance strategies is indeed advantegous (or optimal).

supposed to protect a portfolio from large depreciations while providing unlimited potential for appreciation. The rapid growth of such programs at that time also has to be seen in the light of portfolio insurance firms aggressively marketing dynamic strategies as a risk management device. These firms specialized in selling dynamic strategies as an alleged substitute for real derivative securities.[31] Lessons learned from the crash teach that synthetic derivatives, in contrast to their real counterparts, may fail to fulfill their promises in practice. One major reason is that typical hedge programs do not incorporate that they themselves, if implemented on a large scale, can cause market conditions they are intended to protect against.

Due to the absence of outstanding news about fundamentals, the PRESIDENTIAL TASK FORCE ON MARKET MECHANISMS concentrated on internal market factors, such as portfolio insurance, program trading and stock index arbitrage, during its investigation of the 1987 crash.[32] The qualitative and quantitative findings of the PRESIDENTIAL TASK FORCE ON MARKET MECHANISMS are summarized in GAMMILL and MARSH (1988).[33]

From table 1 in GAMMILL and MARSH (1988, 29-30), one can calculate the proportion of total daily selling volume induced by portfolio insurance programs. In the stock market, that proportion had reached 15.6% on October 19 as seen in table 1.1.

Table 1.1: Selected stock market figures in October 1987.

	stock market sell volume [mUSD]	*portfolio insurance* sell volume [mUSD]	*share* [%]
October 15	4902	257	5.2
October 16	6959	566	8.1
October 19	11197	1748	15.6
October 20	9594	698	7.3

Due to considerable transaction cost savings, portfolio insurers mainly used stock index futures to carry out necessary hedge adjustments. Therefore, it is not surprising that the figures for the futures markets are even more persuasive. The share of portfolio insurance induced selling in the futures markets reached peaks of over 27% on October 19 and October 20.

[31] See JACOBS (1999) for more background information on the role of the purveyors of portfolio insurance in promoting dynamic trading strategies to different groups of investors.

[32] Refer to the survey article of CANINA and FIGLEWSKI (1995) for an introduction to program trading and stock index arbitrage.

[33] The final recommendations (e.g., circuit breakers) are documented in GREENWALD and STEIN (1988).

Corresponding data is found in table 1.2.

Table 1.2: Selected futures market figures in October 1987.

	futures market sell volume [mUSD]	*portfolio insurance* sell volume [mUSD]	*share* [%]
October 15	12655	968	7.6
October 16	15347	2123	13.8
October 19	14801	4037	27.3
October 20	10152	2818	27.8

Estimates for the proportion of market capitalization that was subject to portfolio insurance in 1987 range from only 2 to 3%.[34] Considering these figures, the fast rise and, in particular, the eventual high level of the portfolio insurers' share in total selling volume is surprising. What we learn from this is that even if the market share of dynamic hedgers is relatively low, positive feedback can push their relative share in total daily trading volume to extraordinary levels.

The near-collapse of LTCM

Although the stock market crash of 1987 is outstanding in different respects, the world financial crisis in 1998 exhibits similar characteristics.[35] The striking point about the crisis in 1998 is that it was a world-wide crisis with fragile economic environments particularly in Asia, Russia and Latin America. An often cited victim of the 1998 turmoil is LTCM. Among its most prominent founding partners are Myron Scholes and Robert Merton.[36] The company's main investment vehicle, the Long-Term Capital Portfolio, classifies as a so-called *hedge fund*.[37] In their final report, governmental investigators of the President's Working Group on Financial Markets point out that "LTCM sought to profit from a variety of trading strategies, including convergence trades and dynamic hedging." (page 60). Additionally, the fund used extreme leverage to boost its performance.

[34] Genotte and Leland (1990, 999) report in footnote 2: "Best estimates suggested $70-100 billion in funds were following formal portfolio insurance programs. On a pre-crash total stocks value of about $3.5 trillion, this represents 2-3 percent. Of course, informal hedging strategies as stop-loss selling may have amounted to considerably more than this ..."

[35] See, for instance, Nussbaum (1998) for a commentary on the state of the world economy at that time.

[36] A case study of the LTCM case may also be found in Coy, Woolley, Spiro, and Glasgall (1998).

[37] The report of the President's Working Group on Financial Markets (1999) offers a detailed description of such funds and their favorite investment practices.

After huge losses of 1.8 billion USD in August 1998, its leverage ratio rose dramatically from 54 to 166.[38] By the end of August, 125 billion USD in assets were supported by a capital base of 2.3 billion USD. The capital base shrunk further to 600 million USD by mid-September, supporting balance sheet assets worth about 100 billion USD.[39]

In September 1998, LTCM faced financial markets characterized by a high volatility making hedge programs demand rapid and huge portfolio rebalances. Furthermore, the drying up of liquidity in crucial markets and the 'flight to quality' accelerated as a consequence of even worse economic conditions world-wide. The PRESIDENT'S WORKING GROUP ON FINANCIAL MARKETS has found that the hedge programs that LTCM was following at that time demanded huge hedge adjustments. In fact, they would have exceeded by far the maximum of what the different markets would have been able to absorb. As a result, LTCM could not execute necessary trades. LTCM was caught in a liquidity trap. Fears that the stability of the global financial system could become even more fragile made major financial institutions - under the lead of the Federal Reserve Bank - recapitalize the fund in September 1998. In the end, this step successfully prevented the bankruptcy of LTCM with potentially prolific consequences for the global financial system.

It is worth pointing out that the main difference between the stock market crash of 1987 and the crisis in 1998 is the manner in which dynamic hedging became market influencing. In 1987 it was the simultaneous use of portfolio insurance programs by many market participants. In contrast, the near-collapse of LTCM in 1998 was primarily caused by the high leverage that LTCM built up. Besides this difference the absolute amount of money being dynamically hedged is nonetheless comparable. The severity of the LTCM case is illustrated by the fact that in 1998 assets of LTCM alone totalled 125 billion USD compared to 100 billion USD subject to portfolio insurance schemes in 1987. It is also noteworthy that LTCM's off-balance volume in derivatives contracts amounted to incredible 1400 billion USD in notional value.[40]

This finishes our general discussion of dynamic hedging so that we can now proceed to the subsequent section with an outline of the plan for the remainder of this thesis.

[38] The *leverage ratio* gives the relationship between the capital base and balance sheet assets.

[39] See JACOBS (1999, 285).

[40] See PRESIDENT'S WORKING GROUP ON FINANCIAL MARKETS (1999, 62).

1.4 Organization of the thesis

After having outlined the organization of the thesis, some notes on notation will be given. The basic structure of the thesis is as follows. The thesis is divided into three parts. Part I "Introductory Part" is comprised of two chapters (1 and 2) with introductory and mostly informal material. Part II "Theoretical Foundations", which is comprised of two chapters (3 and 4) as well, introduces basic theoretical concepts applied in part III. Part III "Applications" applies the tools developed in part II to three different economic settings. Each setting is treated in a separate chapter (5, 6 and 7). In the sequel, we give an overview of how the single chapters are structured.

Every chapter (with chapter 1 being the exception) is accompanied by an introduction and a summary of the central results elaborated on within the respective chapter. To begin, chapter 2 of part I surveys work which is closely related to ours. The chapter is intended to introduce the reader to recent work about dynamic hedging in imperfectly liquid markets. While these studies fall into the much broader category of noise trader studies, we confined ourselves to studies that explicitly focus on dynamic hedging. Before going into detail, chapter 2 characterizes similarities between the models proposed in the surveyed articles. Three common model features are identified and discussed. This order helps us to concentrate on results rather than on model features when discussing the articles themselves. In the actual survey, we distinguish between articles that assume complete and symmetric information and those that assume incomplete and / or asymmetric information. The exposition in chapter 2 is mostly informal even though selected results are stated formally when appropriate. The chapter offers a brief look towards application as well.

While preparing this thesis, we tried to keep it as self-contained as possible. Therefore, in part II of the thesis we introduce fundamental methods of financial economics first and develop a general model framework in a second step. All subsequent analyses are then embedded in this general framework. The same holds true for a number of basic definitions and central results which are given in a form applicable to the general framework. The definitions and results can then be securely applied in the specialized settings since they are stated for the general case. This approach prevents us from repeating definitions and results (or from simply omitting them as an 'efficient' alternative).

Chapter 3 of part II reviews selected aspects of modelling uncertainty in a financial markets context. It starts by introducing several probabilistic concepts (e.g., probability space, stochastic process, martingale process) that are capable of capturing basic notions of uncertainty in a financial market.

The main outcome here is a formal model of an economy with uncertainty. The chapter then proceeds with an examination of decision making under conditions of uncertainty. Here, we sketch out the (objective) expected utility approach as well as define common measures of risk aversion. The chapter is rather short and to the point but references are given that can be consulted as necessary.

The basic concepts of chapter 3 are utilized in chapter 4 to develop a general model for a financial market. The main ingredients of this market model are an economy with uncertainty, a time horizon for the economy to exist, a set of securities traded in the financial market and finally a 'set' of agents populating the economy and interacting with each other in the market. The formulation of the market model is general enough to incorporate a great variety of other models [e.g., the COX, ROSS, and RUBINSTEIN (1979) model]. Within this general framework, there is a discussion of the martingale approach to finance. In this respect, chapter 4 sets out basic martingale methods and states powerful results for tackling various problems in financial economics. The chapter culminates in what is called the Fundamental Theorem of Asset Pricing. This theorem postulates the equivalence between the absence of arbitrage, the existence of a certain probability measure, the existence of a (linear) price system for contingent claims and finally the existence of a solution to the problem of an expected utility maximizing agent. Instead of giving proofs for single results, we decided to give detailed references and to include a number of examples to illustrate the application of the results. Throughout chapter 4 we impose the standard assumptions.

Having laid sound theoretical foundations in part II, part III focuses on their application to three different economic settings. Chapter 5 investigates the binomial model as originated by COX, ROSS, and RUBINSTEIN (1979). The model is treated as a special case of the general model presented in chapter 4. The standard assumptions, and notably those of perfectly liquid and complete markets, remain in force there. After some preliminary economic considerations, the binomial pricing formula is reproduced. We then briefly contrast the approach of COX, ROSS, and RUBINSTEIN (1979) with the one of BLACK and SCHOLES (1973). Our objective in doing this is to introduce a tool that enables appealing graphic illustrations of results derived later in the chapter. The emphasis in chapter 5 lies on the characterization of dynamic hedging strategies in the COX / ROSS / RUBINSTEIN model. In particular, we show that dynamic hedging strategies for arbitrary contingent claims with convex payoffs generate positive feedback. To our knowledge, the proof of this result which mostly relies on martingale techniques seems to be new. Utilizing the BLACK / SCHOLES formula, we provide illuminating graphic examples for the general positive feedback result. Considering the analysis of dynamic hedging, chapter 5 sets the stage since it explores dynamic hedging in a setting characterized by the standard assumptions.

After all, the remaining two chapters of part III, namely chapters 6 and 7, contain an analysis of dynamic hedging in imperfectly liquid markets. The models proposed therein are characterized by the fact that we drop the assumptions of *perfectly liquid* and *complete markets*. Since chapters 6 and 7 are structured along similar lines, we will discuss them jointly in the following text.

In contingent claim pricing models relying on the standard assumptions, price processes of securities are assumed to be given exogenously. The main differentiating feature of the models found in chapters 6 and 7 is that price processes are derived from general equilibrium reasoning. In other words, the determination of price processes is endogenized. We are convinced that only a general equilibrium approach is suited to rigorously analyze the impact of dynamic hedging on financial markets. A partial equilibrium approach as the one outlined in section 1.3 is only of limited use when assessing economic implications of dynamic hedging. DUFFIE and SONNENSCHEIN (1989, 567) argue in a similar direction, thereby underpinning our choice:

> "Although, for certain markets, it is possible to explain how price responds to smaller parameter changes with partial equilibrium reasoning, few economists would contend that this method is adequate when economies are disturbed in a major way."

Our approach to analyzing dynamic hedging in imperfectly liquid markets is roughly as follows. We consider a general equilibrium model where we allow a fraction of the whole population to act irrational, which means in this context that they trade on noise rather than on new information. Accordingly, the population divides into two groups: *noise traders* who dynamically hedge given contingent claims and *rational, expected utility maximizing agents*. We will simply refer to these groups as the *hedgers* and the *non-hedgers*, respectively. In equilibrium, non-hedgers set security prices by solving their expected utility maximization problem. More precisely, prices are set such that non-hedgers optimally take security positions that clear the markets. The basic ideas behind this equilibrium concept have already been presented in a simple fashion in section 1.3. In summary, the main feature of our equilibrium approach is that in equilibrium security prices are determined by two factors: *fundamentals* and *aggregate hedge demand*. Standard equilibrium models typically take into account the first factor only, thereby carelessly neglecting the importance of the second one.

In chapter 6 as well as chapter 7, the existence and uniqueness of a general equilibrium is established. Our assumptions about the market models cause the market model in chapter 6 to be complete in equilibrium and the market model in chapter 7 to be incomplete. An analysis of dynamic hedging in a setting where markets are a priori incomplete is new and represents one of our main contributions to the existing literature.

Drawing on the existence and uniqueness results, we carry out a comparative statics analysis to assess the impact of dynamic hedging on security prices and particularly on volatility. Due to market completeness in chapter 6 and market incompleteness in chapter 7, the dynamic hedging strategies followed by the hedgers necessarily differ in both settings. In chapter 6, carefully chosen examples demonstrate that recently published results are not as robust as one might think. Typically, theoretical studies on the impact of dynamic hedging focus on volatility because it is widely considered to be good indicator of market stability. Even though there exists almost overwhelming (theoretical) evidence that positive feedback trading induced by dynamic hedging increases volatility and thereby destabilizes markets, this is not true without further qualification. The examples constructed in chapter 6 show that positive feedback trading by hedgers can also decrease volatility. Similarly, negative feedback trading may also increase volatility. The explanation we offer is based on arguments regarding the liquidity of the financial market. It turns out that in such a context the elasticity of the non-hedgers' demand function is indeed an appropriate measure for market liquidity.

Chapter 7 conducts numerical computations that are intended to give a flavor of how dynamic hedging perturbs the process of the underlying in our model. In particular, chapter 7 contains data showing the dependence of the observed volatility effects on the market weight of hedgers and the degree of the non-hedgers' risk aversion. Computations for European put and call options reveal that the impact of dynamic hedging on volatility is more amplified the higher the market weight is of hedgers and the higher the degree is of the non-hedgers' risk aversion. Comparisons between the complete markets setting of chapter 6 and the incomplete markets setting of chapter 7 suggest that the effects in the former setting are stronger.

We conclude with some remarks.

Some notes on notation

$\mathbb{N}$ denotes the *natural numbers* $\{1, 2, 3, ...\}$, $\mathbb{R}$ the *real line*, $\mathbb{R}_+$ *non-negative real numbers* and $\mathbb{R}_{++}$ *strictly positive real numbers*. If a and b are two real numbers, we define intervals by,

$$[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\},$$

and,

$$]a, b[= \{x \in \mathbb{R} : a < x < b\}.$$

$]a, b]$ and $[a, b[$ are accordingly defined. '$\equiv$' means *equal by definition.* '$\forall$' is the universal quantifier and means *for all.* '$\emptyset$' denotes the *empty set.* 's.t.' is short for *subject to.* Whenever possible, probability measures are typeset in 'bold' (e.g., $\mathbf{P}$ or $\mathbf{Q}$), sets are typeset in 'blackboard bold' (e.g., $\mathbb{A}$ or $\mathbb{T}$), and systems of sets, like $\sigma-$algebras, for instance, are typeset in 'calligraphic' (e.g., $\mathcal{F}$ or $\mathcal{M}$). $|\mathbb{E}|$ denotes the number of elements of a finite set $\mathbb{E}$. For a given set $\mathbb{E}$, the indicator function is defined by,

$$\mathbf{1}_{\mathbb{E}}(a) \equiv \begin{cases} 1 \text{ if } a \in \mathbb{E} \\ 0 \text{ if } a \notin \mathbb{E} \end{cases}.$$

For a given function f, we sometimes denote the first derivative by f' and the second derivative by f''. As common, we define,

$$n! \equiv \prod_{i=1}^{n} i.$$

Apart from the aforementioned notational conventions, we use fairly standard notation. However, detailed comments on the notation will be given if necessary.

Chapter 2

Related work

2.1 Introduction

Conclusions drawn from market observations that dynamic hedging may increase market volatility and even cause stock market crashes have stimulated a lot of theoretical studies investigating dynamic hedging.[1] Such investigations typically take place in a partial or general equilibrium framework, thereby mimicking imperfectly liquid markets. Two basic approaches can be identified in this strand of literature. One approach is to focus on technical properties of dynamic hedging strategies (e.g., positive feedback) and to assess what impact these strategies have on equilibrium prices in a complete information setting. Considering equilibrium prices instead of exogenously given prices is equivalent to replacing the assumption of perfectly liquid markets by the much milder assumption of imperfectly liquid markets. Studies that fall into this class are the focus of section 2.3.

The other basic approach is to assume that market participants are not aware of the extent to which dynamic hedging strategies are implemented. This, in turn, is equivalent to relaxing the assumption of complete information and / or symmetric information of standard contingent claim pricing models. The unawareness of agents regarding dynamic hedging potentially leads to misinterpretations of observed trading activity. For example, uninformed agents may interpret trading activity induced by dynamic hedging as informed trading and therefore jump on the bandwagon. Furthermore, incomplete or asymmetric information can cause coordination problems in providing sufficient liquidity to the market, which may eventually lead to a further decline in market liquidity. Section 2.4 surveys studies stressing this

[1] Schwert (1990), for instance, is a comprehensive empirical study of volatility-related issues which is inspired by the stock market crash of October 1987.

information problem.[2]

Although the scope of the articles surveyed in this chapter and the models proposed therein are rather diverse, the different models nevertheless share several basic features. This makes it worthwhile to discuss in advance the most important ones in section 2.2. Having laid this foundation, the detailed discussion taking place in sections 2.3 and 2.4 can then concentrate on highlighting results. Section 2.5 points to some areas of application while section 2.6 finally summarizes the most important aspects from our point of view.

2.2 Common model features

We can identify three common features in which the models of the surveyed articles share. First, almost all market models entail two *securities*: a *risky* one, e.g., a stock, and a *risk-less* one, e.g., a bond. The price processes are such that they complete markets a priori, i.e., every contingent claim is attainable via an appropriate trading strategy in the two securities. In particular, this implies that arbitrary derivative securities can be dynamically hedged. However, the models differ considerably in respect to the number of trading possibilities. Some authors assume that there are only finite trading possibilities, or equivalently, that time is discrete. Others assume that trading is continuously possible, or equivalently, that time passes continuously. Whether discrete time or continuous time, the restriction of two securities enables the comparison of the obtained results with those obtained in standard models like Cox, Ross, and Rubinstein (1979), in the case of a discrete time analysis or Black and Scholes (1973), in the case of a continuous time analysis.

Second, the authors generally assume that *two groups of agents* trade in the security markets: *hedgers* and *non–hedgers*.[3] Sometimes the group of non-hedgers may again divide into several sub-groups. Non-hedgers are generally modelled as rational, expected utility maximizing agents. In contrast, hedgers implement dynamic hedging programs in order to hedge derivative securities, whereby their motivation to do so is typically exogenously given. Two basic approaches are applied to model the trading behavior of

[2] We should note that it is rather difficult to make a clear distinction between complete and symmetric information models and incomplete and / or asymmetric information models. However, the grouped articles seem to be sufficiently homogenous in terms of emphasis.

[3] The authors almost always use names for the two groups that deviate from those proposed here. We will nevertheless keep on using the expressions *hedgers* and *non-hedgers*. This will hopefully facilitate to recognize similarities as well as differences in the single models.

the hedgers. One is to postulate that the hedgers act like automata, i.e., they only execute a given hedge program. The other is to presume that the hedgers maximize their expected utility while facing an additional constraint. This last method applies well to portfolio insurance. In such a case, a floor constraint for end of economy wealth is imposed for the hedgers. Independent of the differences in modelling the hedgers, trading by the hedgers always produces positive feedback in the market, i.e., they buy when prices climb and sell when prices slip.

Third, and probably the most important common feature, is that the market of the risky security is *imperfectly liquid* in the models. As a consequence, large enough trades can influence the price of the risky security, particularly those for the purpose of hedging. However, the ways of modelling imperfectly liquid markets are numerous, and in fact spanning a wide range. On the one hand, for example, some authors propose partial equilibrium frameworks where the demand of the hedgers and the reaction of the security prices to their trades are assumed to follow a certain rule. On the other hand, some authors consider pure general equilibrium frameworks where the demand functions of all agents are derived endogenously, and prices are set such that markets clear.

In light of the above said, we can summarize the different aspects as follows: Two types of agents, hedgers and non-hedgers, interact in a priori complete but imperfectly liquid securities markets in which two securities, a risky one and a risk-less one, are generally available.

2.3 Complete information models

2.3.1 Brennan and Schwartz (1989)

Inspired by the events surrounding the stock market crash of October 19, 1987, Brennan and Schwartz (1989) analyze the impact of portfolio insurance on stock market prices. They embed their formal analysis of portfolio insurance in a general equilibrium model where hedgers and non-hedgers operate in a stock and a bond market. They interpret the stock as the market portfolio. The non-hedgers, exhibiting constant relative risk aversion (CRRA)[4], maximize their expected utility of end-of-period consumption in contrast to the hedgers, who act like automata following a portfolio insurance strategy. Since they are representative in a certain sense, the non-hedgers

[4]Throughout this chapter we will frequently encounter theoretical concepts not yet formally introduced. Most of them - like constant relative risk aversion - will be introduced in part II. These are circumstances we can hardly avoid.

set the prices in the economy. The authors assume that all agents are aware of the implementation of portfolio insurance itself and the extent to which this takes place.[5]

Since this assumption implies rather liquid markets, they observe rises in the volatility of only 1% when the market weight of the hedgers is 5% and the degree of CRRA of the non-hedgers is 2. For these parameter values, they calculate an increase of approximately 4.6% for the market risk premium.[6] Furthermore, they find that the costs for implementing portfolio insurance rise with increasing adoption of such strategies, which means that markets are no longer linear. In linear markets, if the price for a synthetic put is $a > 0$ then the price of the put is still a even if the demand for it doubles, for instance. Instead, the model of Brennan and Schwartz (1989) predicts a price of $a + c$ for the synthetic put, where $c > 0$, when the demand for portfolio insurance doubles.

For the general case, Brennan and Schwartz (1989) demonstrate that the market volatility σ in the presence of hedgers satisfies,

$$\sigma_t(\eta_t) = \underbrace{\frac{\eta_t}{S_t} \cdot \frac{\partial S_t}{\partial \eta_t}}_{(A)} \cdot \sigma^*, \tag{2.1}$$

where η_t is the fundamental state variable, t denotes time, S_t is the price of the market portfolio in the presence of hedgers (in terms of the numeraire) and σ^* is the reference or input volatility.[7] The term (A) may be interpreted as the correction term accounting for feedback; it is greater than or equal to 1. In the absence of portfolio insurance, $(A) = 1$ and $\sigma = \sigma^*$ as desired. As a result, dynamic hedging destabilizes financial markets.

[5] "The strategy of the portfolio insurer and his resulting payoff function are known to all market participants." Brennan and Schwartz (1989, 458).

[6] The *market risk premium* is defined as,

$$\mu_M - r_f,$$

where μ_M is the expected rate of return of the market portfolio, r_f is the risk-less interest rate. The *market price of risk* (or the Sharpe *ratio*) is defined as,

$$\frac{\mu_M - r_f}{\sigma_M},$$

where σ_M is the volatility of the market portfolio ($\equiv$ standard deviation of its rate of return). Refer, for example, to Jones (1998, 230-232).

[7] Compare equations (10), (12) and (14) in Brennan and Schwartz (1989, 459-460).

2.3.2 Donaldson and Uhlig (1993)

The study of DONALDSON and UHLIG (1993) investigates the effect large hedgers have in comparison to an atomistic group of hedgers who they call 'portfolio insurers'. In their model allowing trade in a stock and a bond, non-hedgers having constant absolute risk aversion (CARA) preferences interact with hedgers following a simple stop-loss trading rule. For this setting, they show that,

> "... the existence of atomistic portfolio insurers increases the variance of possible equilibrium prices (i.e., volatility) and can lead to situations in which there are many potential equilibrium prices for a single set of fundamentals." DONALDSON and UHLIG (1993, 1943).

The trading rule the hedgers implement requires them to invest all available wealth in the stock if the stock price is above a certain level and to invest it in the bond if the stock price is below that threshold.[8] Their model has two different, stable equilibria. They interpret one of them as the undesirable or *crash* equilibrium. In this context, the hedgers' trades may cause a shift from one equilibrium to the other equilibrium. Thus, a crash occurs according to DONALDSON and UHLIG'S (1993) interpretation.

However, if there are instead large hedge firms who act on behalf of a number of individuals, the effects on volatility are smoothed since the probability for a crash decreases. This holds even if these firms attract more agents than would otherwise follow a hedge strategy. The intuition DONALDSON and UHLIG (1993) provide for this result is that a large hedge firm that is aware of its impact on prices may reconsider planned sell orders whereby it may avoid a market crash. Stated differently, a hedge firm, aware of its market power, may adjust its trading activity to 'market realities' so that the stock price does not jump down from the desired equilibrium to the undesired one.

2.3.3 Jarrow (1994)

In a partial equilibrium model with a stock and a bond, mainly based on JARROW (1992), JARROW (1994) investigates dynamic hedging by a single large hedger, i.e., someone whose trades may affect equilibrium prices. The large hedger implements dynamic hedging strategies for derivatives, in particular for call options and forwards. The reaction function of the security prices to the actions of the large hedger are given exogenously. JARROW (1994)

[8]Section 14.3 of HULL (1997) examines this kind of hedging scheme.

explains that his assumptions with respect to the reaction function are consistent with several equilibrium models. Small hedgers act as price-takers, making up for the rest of the population.

In this setting, JARROW (1994) shows that the introduction of option markets may produce *market manipulation strategies*, even if they did not exist before. He defines market manipulation strategies as arbitrage opportunities for the large hedger. However, applying a condition he calls *synchronous markets* effectively excludes such strategies. Synchronous markets mean that it does not matter whether stock holdings of the large hedger are direct (in the stock) or indirect (via an option on the stock). JARROW (1994) identifies this condition to be equivalent to the *no arbitrage* condition in perfect competition models. With this condition and under a complete information assumption, he is able to derive an option pricing theory for small hedgers in a market with a large hedger. JARROW (1994, 252) comments on his pricing theory:

> "This theory has the important result, under a common knowledge assumption, that to a price taker, the standard binomial model still applies, but with a random volatility."

But if the small hedgers have only incomplete information regarding the large hedgers' actions and the reaction function, they may fail to "synthetically construct the call options" (page 258). JARROW (1994, 242) points out that his pricing theory may help to explain market anomalies as well:

> "Due to the reason for the random volatility, this new theory has the potential to explain some previously puzzling empirical deviations from the standard Black-Scholes formula."

Although DONALDSON and UHLIG (1993) and JARROW (1994) both model large hedgers having the ability to influence the market, the pictures painted of these hedgers differ significantly. Whereas DONALDSON and UHLIG (1993) suppose that the economy as whole can benefit from large hedgers being able to control their trades, JARROW (1994) presumes that large hedgers try to manipulate markets for selfish reasons whenever possible.

2.3.4 Balduzzi, Bertola, and Foresi (1995)

BALDUZZI, BERTOLA, and FORESI (1995) also examine an economy in which non-hedgers and hedgers interact with each other. As common for similar studies, these agents can trade a stock and a bond. The non-hedgers set prices in equilibrium taking into account demand shocks caused by the hedgers. Though BALDUZZI, BERTOLA, and FORESI (1995) model an economy with continuous information flow, they assume that hedgers trade at discrete points in time only. Mainly relying on numerical examples, they particularly analyze cases where hedgers trade once. In our view, this is a major drawback of their study since the notion of dynamic hedging gets lost.

BALDUZZI, BERTOLA, and FORESI (1995) show that positive feedback shocks cause the stock price volatility to increase. This is in line with results obtained by other authors. However, they additionally demonstrate that negative feedback shocks caused by the hedgers have a stabilizing effect; such shocks decrease the stock price volatility. Moreover, they report that the observed effects lead to predictability of stock returns which clearly contradicts the EMH. Assuming CRRA preferences for the non-hedgers, BALDUZZI, BERTOLA, and FORESI (1995) find that the effects of feedback trading are stronger the higher the non-hedgers' degree of CRRA is.

2.3.5 Basak (1995)

The focus of BASAK (1995) is similar to the one of BRENNAN and SCHWARTZ (1989). His analysis takes place in the general equilibrium model of LUCAS (1978), with hedgers and non-hedgers both being expected utility maximizing agents having CRRA preferences. These agents maximize their expected utility by trading a large number of available securities. Furthermore, they consume continuously and not, as in BRENNAN and SCHWARTZ (1989), at the terminal date only. BASAK (1995) mainly utilizes the martingale representation methods developed in COX and HUANG (1989). He introduces portfolio insurance via an additional constraint for the optimization problem of the hedgers. This constraint requires that the wealth at a fixed horizon must not fall under a given floor.[9] The model of BASAK (1995) differs significantly in one way from similar studies in that he assumes that the hedge horizon lies before the end of the economy. He thereby allows for discontinuities in securities prices between the hedge horizon and the economy horizon.

BASAK (1995) finds that in his model, portfolio insurance *decreases* the volatility of the underlying securities. The rationale he gives is that securities

[9] GROSSMAN and VILA (1988) show that in complete markets such a problem is equivalent to the problem of an expected utility maximizing agent not facing the mentioned constraint but rather owning an appropriately chosen put option. As is reasonable, the initial endowment of this agent is accordingly reduced by the price of the put option.

A brief formal account of an approach to modelling hedgers similar to BASAK'S (1995) will be given in sub-section 2.3.7 where we discuss the model of GROSSMAN and ZHOU (1996).

must become more favorable to give the risk-averse agents an incentive to absorb the additional supply from the hedgers in some states of the world. In particular, Basak (1995, 1078) notes:

> "In our setup (CRRA preferences), the market price of risk is unchanged across economies, so the only way to achieve favorable risky assets is for the volatility of some risky assets (and hence the market) to decrease. Since the market volatility is decreased, for the market price of risk to remain unchanged the market risk premium must also decrease."

Neither does intuition support the result that positive feedback trading decreases volatility and market price of risk, nor does any other study we know of confirm these results.

2.3.6 Frey and Stremme (1995)

Frey and Stremme (1995) work in an overlapping generations model with trading opportunities in a stock and a bond. The model economy is populated by two types of agents: non-hedgers, today maximizing their expected utility of consumption tomorrow, and hedgers, today hedging a contingent claim payable tomorrow. The authors' focus lies on general dynamic hedging strategies exhibiting the positive feedback characteristic rather than on specific types of positive feedback strategies. Working in continuous time, they derive a stochastic differential equation the equilibrium stock price process must satisfy when hedgers are present. This equation boils down to the standard equation of Black and Scholes (1973) when the hedgers' market weight is zero, indicating consistency with the standard theory.

In contrast to a pure general equilibrium approach, they postulate certain characteristics the demand functions of the agents should satisfy. For example, stock demand functions, derived for the special case where the non-hedgers have CRRA preferences, have the postulated properties. For this special case they prove existence and uniqueness of a general equilibrium. They derive the following expression for the market volatility σ in the economy where hedgers with market weight $\alpha \in [0, 1[$ are active,

$$\sigma = \underbrace{\frac{1 - \alpha \cdot \phi_t(S_t)}{1 - \alpha \cdot \phi_t(S_t) - \alpha \cdot S_t \cdot \frac{\partial \phi_t(S_t)}{\partial t}}}_{(B)} \cdot \sigma^*. \tag{2.2}$$

Here, t denotes time, S_t the stock price at t, $\phi_t(S_t)$ is the number of shares of the stock held at t to dynamically hedge contingent claims and σ^* is the reference or input volatility.[10] One sees that for $\alpha = 0$, $\sigma = \sigma^*$. Moreover, σ is increasing in α, i.e., the more hedgers are active the higher is the volatility in the economy. In other words, the volatility correction term (B) satisfies $(B) \geq 1$, as does the corresponding term (A) [see equation (2.1)] in the model of Brennan and Schwartz (1989).

Frey and Stremme (1995) elaborate that even if in this environment hedging strategies perfectly replicating a derivative security fail to exist, the Black / Scholes-methodology may still be applied. The adjustment to be made is in respect to the input volatility. The Black / Scholes volatility has to be replaced by a so-called *super-volatility*, as defined on page 16 in Frey and Stremme (1995), that leads to a super-replication of the derivative.[11] They prove the existence of a unique super-volatility in their setting.

Using the same parameter values as Brennan and Schwartz (1989), Frey and Stremme (1995) find that their model predicts stronger effects of dynamic hedging on market volatility. For a market weight of hedgers of 5% and a degree of CRRA of 4, they get from numerical simulations that the market volatility rises by 9%, while it rises by only 2% in the Brennan and Schwartz (1989) model for these parameter values. They attribute this to the different modelling of the non-hedgers. Contrary to Brennan and Schwartz (1989) where non-hedgers consume at the terminal date only, non-hedgers here consume continuously, thereby taking "... changes in current prices as signals for future price movements." (page 3). Frey and Stremme (1995, 18) comment on their findings:

> "A comparison with the analysis of Brennan and Schwartz revealed the importance of agents' expectations in determining market liquidity and hence the amplitude of the feedback effect on volatility."

2.3.7 Grossman and Zhou (1996)

The study of Grossman and Zhou (1996) is closely related to those of Brennan and Schwartz (1989) and Basak (1995) since it also analyzes portfolio insurance as a special case of dynamic hedging. Similar to Basak (1995), Grossman and Zhou (1996) model portfolio insurance by an additional (floor) constraint that one group of the agents, the hedgers, faces.

[10] Compare equation (3.15) in Frey and Stremme (1995, 12).

[11] Roughly speaking, a trading strategy ϕ that *super-replicates* a contingent claim is a trading strategy whose payoff $V(\phi)$ dominates the contingent claim's payoff A at maturity, $V(\phi) \geq A$. In contrast, for a trading strategy that *replicates* the contingent claim $V(\phi) = A$ holds. In chapter 4, super-replication is introduced in a rigorous manner.

All agents of their general equilibrium model only consume at the end of the economy, which is in accordance with the modelling of the non-hedgers in Brennan and Schwartz (1989). In the model, agents can trade a stock, interpreted as the market portfolio, as well as a bond. In the analysis of the agents' problems, Grossman and Zhou (1996) mainly utilize martingale methods. Given an initial endowment of W_0, the problem of a non-hedger is to,

$$\max_{W_T} \mathbf{E}_0^{\mathbf{P}}[v(W_T)] \tag{2.3}$$

$$\text{s.t. } W_T = \phi_T^0 + \phi_T^1 \cdot \eta_T \tag{2.4}$$

$$dW_t = \phi_t^1 dS_t^1, \tag{2.5}$$

where T is the economy's terminal date, $\mathbf{E}_0^{\mathbf{P}}$ denotes expectation at time 0 under the given probability measure $\mathbf{P}$, $v(\cdot)$ is the utility function of the risk-averse non-hedger and W_T is his / her time T wealth. ϕ_T^0 denotes the number of units of the bond, serving as the numeraire, held at T. $\phi_t^1, t \in [0, T]$, denotes the number of shares of the stock held at time t and η_T the liquidating dividend of the stock. Finally, S_t^1 is the price of the stock prior to T, i.e., for $t \in [0, T[$. A hedger faces the additional constraint,

$$W_T \geq \varepsilon \cdot W_0, \tag{2.6}$$

with $\varepsilon \in [0, 1]$. In other words, a hedger has to assure that his / her terminal date wealth W_T does not fall under a given floor expressed as a fraction of his / her initial wealth W_0. For the hedgers, $v(\cdot)$ may be replaced in (2.3) by another utility function. The application of martingale methods transforms problem (2.3)-(2.5) into the static problem,

$$\max_{W_T} \mathbf{E}_0^{\mathbf{P}}[v(W_T)]$$

$$\text{s.t. } \mathbf{E}_0^{\mathbf{P}}[\mathbf{L}W_T] = W_0,$$

where $\mathbf{L}$ is the state price density with respect to a $\mathbf{P}-$equivalent martingale measure.[12]

For two specific examples, Grossman and Zhou (1996) find that the presence of dynamic hedging increases both the volatility and the market price of risk. The examples differ only in the specification of the hedgers' utility function. The first example assumes log utility, implying CRRA of 1 for both groups of agents, whereas in the second example the hedgers have

[12] Refer, for instance, to section 3.4 of Baxter and Rennie (1996). In part II of the thesis, we introduce these concepts for a discrete time setting.

CRRA of $\frac{1}{2}$. In these parameterized settings, the aforementioned effects are stronger when bad news arrives and weaker when good news arrives. The intuition behind this is that when market prices slump, thereby approaching the hedgers' target floor, hedgers must sell shares of the stock in order to satisfy constraint (2.6). This additional selling pressure causes prices to further decline, therefore volatility increases. To give the non-hedgers an incentive to absorb the additional stock supply by the hedgers, the attractiveness of the stock must rise, therefore the market price of risk increases. As an aside, GROSSMAN and ZHOU (1996) also show that there is a positive correlation between trading volume and price volatility.

2.3.8 Sircar and Papanicolaou (1997)

Considering the general equilibrium model of FREY and STREMME (1995), SIRCAR and PAPANICOLAOU (1997) ask the question of how the generalized BLACK / SCHOLES model[13] must be adapted to account for feedback from dynamic hedging of call options. They derive for the equilibrium price process the following partial stochastic differential equation,

$$dS_t = \mu_t(S_t, \eta_t)dt + \underbrace{k_t(S_t, \eta_t)}_{(C)} \cdot \sigma_t^*(\eta_t)dw_t, \tag{2.7}$$

where S_t is the stock price at time t, $\mu(\cdot)$ is the drift term, η_t is the (stochastic) income of the non-hedgers at t, $k_t(\cdot)$ respectively (C) is the correction term for the feedback from dynamic hedging, $\sigma_t^*(\cdot)$ is the income process' volatility and w_t is a standard BROWNIAN motion.[14] In the absence of hedging activity with considerable influence on the market, the correction term (C) vanishes, i.e., it equals 1, so that the model boils down to the generalized BLACK / SCHOLES model. A fact we have already observed twice; in the model of BRENNAN and SCHWARTZ (1989) and in the model of FREY and STREMME (1995) [see equations (2.1) and (2.2)]. Here, however, this reduction depends on the crucial assumption that the non-hedgers are endowed with a stock

[13] They "... use the word generalized in the sense that the underlying asset price is a general ITÔ process rather than the specific case of geometric Brownian Motion in the classical Black-Scholes derivation." SIRCAR and PAPANICOLAOU (1997, 5). Refer to chapter 3 of BAXTER and RENNIE (1996) for an introductory overview of stochastic processes in continuous time, such as BROWNIAN motion, and the necessary stochastic calculus tools to handle them. A more comprehensive treatment of this and related issues (e.g., stochastic integration) is found in KARATZAS and SHREVE (1988).

[14] Compare equations (2.9), (2.10) and (2.11) in SIRCAR and PAPANICOLAOU (1997, 8).

demand function of the functional form,

$$\phi(S, \eta) = f\left(\frac{\eta^\gamma}{S}\right),$$

where S is the stock price, η the income, $\gamma \equiv \frac{\sigma^{BS}}{\sigma^*}$ with σ^{BS} being the BLACK / SCHOLES volatility and σ^* being the income volatility, and where f is an arbitrary smooth, increasing function. The reason for this requirement with respect to the demand function of the non-hedgers stems from the fact that the BLACK / SCHOLES setting is compatible with a general equilibrium framework under rather restrictive assumptions only. BICK (1987) studies this topic.

For a market weight of hedgers of 5%, SIRCAR and PAPANICOLAOU (1997) report that the observed volatility in a specific example may rise by up to 12% relative to the benchmark where the hedgers market weight is negligible. Numerical simulations for a set of call options with strike prices being evenly distributed reveal that volatility is highest at the mean of strike prices, amounting there to an 8% increase relative to the benchmark.

2.4 Incomplete information models

KLEIDON (1995) surveys the three articles treated in this section as well. In his survey article about stock market crashes, KLEIDON focuses on the explanatory power of the different models with respect to the stock market crash of 1987. As we will see below, incomplete information models seem well-suited for explaining such extraordinary events as opposed to complete information models.

2.4.1 Grossman (1988)

While its publication date lies after the crash of October 1987, an early version of GROSSMAN'S (1988) article was written before the crash. Nevertheless, GROSSMAN (1988), acknowledging the importance of dynamic hedging strategies in the determination of market prices, explores how the implementation of dynamic hedging programs may influence the market's volatility. In addition, he examines how price jumps (such as crashes) may be explained by the use of these programs while the emphasis, however, lies on the volatility issue. In his article, he stresses the central idea that real derivative securities and their synthetic counterparts are imperfect substitutes:

> "In particular, the replacement of a real security by synthetic strategies may in itself cause enough uncertainty about the price

> volatility of the underlying security that the real security is no longer redundant." GROSSMAN (1988, 276).

GROSSMAN (1988) argues that a real, exchange-traded security carries information that would not be available if one replaces the security by a dynamic hedging strategy. An example he cites, is that the price of a put option reflects the sentiments of investors about future market developments. If the majority of market participants expect price declines, put prices should be high reflecting a high protection demand. Otherwise, he argues, if market participants are rather optimistic, put prices should be low because of weak demand. This information content gets lost when market participants implement dynamic strategies, such as portfolio insurance, rather than buying real options.

GROSSMAN (1988) embeds his analysis in an informal three date model where three groups of agents are active: hedgers, liquidity providers, and buy-and-hold managers. One major shortcoming of GROSSMAN'S analysis is that he, since arguing informally, has to *assume* that the implementation of dynamic hedging strategies increases the market volatility. He further *assumes* that this effect is stronger when the liquidity providers, because of a lack of information, do not commit enough capital in the market to absorb the hedge-induced trades. Moreover, GROSSMAN (1998, 297) infers that,

> "[w]ith incomplete information about portfolio insurance usage, [the fraction of hedgers] f should be modeled as a stochastic process. ... The consequent stochastic volatility will make put options no longer redundant."

2.4.2 Genotte and Leland (1990)

The aim of the study of GENOTTE and LELAND (1990) is to answer the following questions relating to the 1987 stock market crash:

> "1) How can relatively small amounts of hedging drive down prices significantly?
> 2) Why didn't stock prices rebound the moment such selling pressure stopped?" GENOTTE and LELAND (1990, 1000).

Whereas the majority of the aforementioned studies are mainly concerned with volatility alone, GENOTTE and LELAND (1990) explicitly seek to explain why the crash of October 1987 could occur provided dynamic hedging was its primary cause.

Four types of agents populate the model economy GENOTTE and LELAND (1990) consider: hedgers, uninformed traders, supply-informed traders and price-informed traders. All but the hedgers exhibit CARA preferences. The signals the informed traders receive are imperfect or even poor. GENOTTE and LELAND (1990, 1008) note that "[t]he key to the stability of markets is the extent to which hedging strategies are observed by investors." The rationale they give is that "[t]he relative proportion of investors who are informed versus uninformed is a key determinant of market liquidity." (page 1002). The former aspect relates to incomplete information in the model economy while the latter aspect concerns an asymmetric information problem.

Using examples, they analyze three cases: one where the hedge-induced supply is fully anticipated, one where it is partially observed and one where it is totally unobserved. Since the first two cases both lead to one stable equilibrium only, crashes can only occur in the third case which includes two stable equilibria. In this case of particular interest, the market may crash because of hedge-induced selling if the stock price reaches a certain threshold. After the crash, when the market is in the 'low-price' equilibrium, the market remains there if it is stable enough, which is in fact the case in their model. This can explain why it took stock markets many months to recover after the 1987 crash. Furthermore, the model of GENOTTE and LELAND (1990) predicts that the volatility in the new equilibrium is lower than before the crash, which is a result that is supported by empirical observations as well.[15]

2.4.3 Jacklin, Kleidon, and Pfleiderer (1992)

In similar vein, JACKLIN, KLEIDON, and PFLEIDERER (1992) also seek to explain the stock market crash of October 1987. Notably, their analytical focus differs from those of GROSSMAN (1988) or GENOTTE and LELAND (1990):

> "... we focus not on the potential liquidity problems caused by coordinated selling - which we acknowledge as real - but rather on the inferential problem caused prior to such selling. ... We are particularly interested in the problem caused for those investors who rationally condition their demand on the demands of others because they are imperfectly informed." JACKLIN, KLEIDON, and PFLEIDERER (1992, 36).

In their analysis, JACKLIN, KLEIDON, and PFLEIDERER (1992) mainly rely on numerical simulations for a model in which three groups of agents

[15] SCHWERT (1990, 77) writes: "Stock volatility rose dramatically during and after the 1987 crash, but it returned to lower, normal levels unusually quickly."

(hedgers, informed traders, and liquidity traders) interact with a market maker making markets in a single stock. They demonstrate that the underestimation of dynamic hedging in their model leads to greater prices than implied by fundamentals. Market participants who learn over time about dynamic hedging usage gradually reevaluate their previous inferences. When the extent of dynamic hedge programs is fully revealed, prices may abruptly fall, hence a market crash occurs.

This concludes the actual survey. The next section is concerned with possible areas of application for the models just discussed.

2.5 A brief look towards application

In this section, we briefly discuss potential areas of application of the models proposed in the surveyed articles. In principle, three main areas can be distinguished. First, these models may help to explain empirical observations. Second, they can help to address regulatory issues. Third and finally, they may help to improve existing financial models with regard to their applicability to real world problems.

All but one of the studies surveyed conclude that dynamic hedging strategies *increase market volatility.* Empirical evidence that there is indeed excess volatility in markets is cited in section 1.2. A more general conclusion that can be drawn from this result is that market volatility is dependent on the fraction of the market portfolio subject to dynamic hedging. Naturally assuming that this fraction is uncertain or stochastic, these studies also deliver an explanation for why market volatility is rather *non-deterministic,* i.e., uncertain or stochastic, than deterministic as assumed in standard models. Uncertain or stochastic volatility in turn generally implies that markets are incomplete. Moreover, models that incorporate feedback effects from dynamic hedging may also deepen our understanding of certain empirical observations called *market anomalies.* Grossman and Zhou (1996), for instance, show that their model is consistent with the volatility smile.[16] They note:

> "... the existence of the volatility smile in the options market is evidence that the options market has priced in the equilib-

[16] The *volatility smile* refers to the phenomenon that option prices of options on the same underlying but with different strike prices imply different volatilities for the underlying. A graph of the implied volatilities against the strikes often looks like a 'smile' which is where the name comes from. Of course, the volatility smile is inconsistent with the Black / Scholes theory. See section 19.7 in Hull (1997).

> rium implications of portfolio insurance." GROSSMAN and ZHOU (1996, 1398)

Having analyzed what impact dynamic hedging has on market prices, the next step is to ask what the consequences are for market participants. However, GROSSMAN (1988, 293) warns:

> "..., even if dynamic hedging strategies have contributed (or will contribute as their importance grows) to stock price volatility, it does not follow that this is, in net, socially harmful or worthy of regulation. To say that the use of a strategy imposes costs hardly implies that these costs outweigh their benefits."

A first step could therefore be to assess *welfare implications* of dynamic hedging. A natural next step would then be to ask whether dynamic hedging is worthy of *regulation* or not. Also a possible field of application is the *estimation of dynamic hedging activity* in securities markets which, for example, may be of interest to a regulatory authority. To conduct such an estimation, one has to invert the procedure of assessing the impact of dynamic hedging on the market. The question then is: Given an observed market volatility, what does this imply for the market weight of hedgers?

Another important area of application is the *pricing and hedging of derivatives* in imperfectly liquid markets. FREY (1996) derives a pricing and thereby a hedging theory for a large hedger in such markets. JARROW (1994) considers the opposite side: small hedgers facing a large hedger. As already mentioned above, JARROW (1994) proves that the binomial option pricing model still applies for small hedgers under a common knowledge assumption but with stochastic, or more exactly, level-dependent volatility. Similarly, SIRCAR and PAPANICOLAOU (1997) derive a pricing theory for small hedgers in the presence of feedback. In a sense, their model is a generalization of the BLACK / SCHOLES / MERTON theory to imperfectly liquid markets. Due to recent successes of *stochastic volatility models* in explaining alleged market anomalies and overcoming some known shortcomings of the standard models, such models become increasingly popular in the derivatives industry. That is why SIRCAR and PAPANICOLAOU (1997) suggest that considering feedback effects from dynamic hedging in these models can further enhance their applicability.

2.6 Summary

As it becomes evident in the studies surveyed in this chapter, no matter whether information is complete or incomplete, the qualitative predictions concerning the impact of dynamic hedging on financial markets are generally the same. An often reported result is, for example, the increase in the market volatility, or as it may be interpreted, a destabilization of financial markets. However, the effects predicted are intensified when information is incomplete, typically due to a further decrease in market liquidity. That dynamic hedging increases market volatilities comes from the positive feedback it poses on the markets: price rises are amplified by additional buy orders from hedgers, price falls are accelerated by additional sell orders.

A puzzle that remains unresolved is how positive feedback hedging can decrease volatility as found in Basak (1995). Moreover, so far it is not clear what impact dynamic hedging has when markets are incomplete a priori. In such markets, hedge strategies are no longer uniquely determined. This suggests that dynamic hedging in such a context may possibly produce different results compared to those obtained in complete markets. We will tackle both points in part III of the thesis.

Part II

Theoretical Foundations

This part of the thesis is intended to provide a review of three different topics in financial economics: modelling uncertainty, decision making under uncertainty and the martingale approach. If done rigorously, each of these topics deserves a book-length treatment. For this reason, we cannot and do not even attempt to be comprehensive in exploring these areas of financial economics. We can, however, take a glance at some basic methods, helpful techniques and central results which will be applied in part III of the thesis. This helps to keep the thesis as self-contained as possible.

Our guiding principle for the selection of the topics covered in this part was the importance of the different economic theories for the applications in part III. By all means, the theory of decision making under uncertainty and the martingale approach are central there. Basic principles in modelling uncertainty are of course necessary prerequisites for the other two fields. At the end of part II, we will be equipped with a basic market model suited to include as special cases a great variety of common market models. Furthermore, we will have available a tool-set we can use to attack diverse problems typically arising in situations under uncertainty.

The exposition in chapters 3 (uncertainty and decision making under uncertainty) and 4 (martingale approach) is short and to the point. Results are stated without proof but detailed references are given in each case. Instead of giving proofs we decided to include several examples that illustrate the application of central results. Our hope is that the examples are a better guide to the ideas behind the different results than the corresponding proofs, which are rather technical in general.

Chapter 3

Uncertainty - A quick review

3.1 Introduction

Financial economics is mainly concerned with decisions under conditions of uncertainty. Without uncertainty, the field would presumably not exist at all. Therefore, in this short chapter we develop a model for an economy with uncertainty and briefly review one possible approach to decision making under uncertainty.

The model introduced in section 3.2 builds the basis for all subsequent analyses. Frankly, the concepts introduced there are standard probabilistic concepts which we interpret from an economic point of view. A central concept tackled in this section is the concept of a martingale. In a sense, this represents the prelude to the next chapter where we explore the martingale approach to financial economics. Since the body of literature that deals with the topics presented in section 3.2 is extremely large, we only give selected references. BAUER (1990) is rather comprehensive with respect to the topics covered there. He has a strong focus on theoretical aspects only. In addition, the book by BROCK and MALLIARIS (1982) provides applications to economics and finance. It is, however, not as comprehensive in theoretical terms as BAUER (1990). Chapter 1 of KARATZAS and SHREVE (1988) contains a very compact presentation of the relevant material. Chapter 3 of PLISKA (1997) may also be consulted. The strength of his exposition is that economic considerations motivate the use of mathematical concepts. Moreover, the measure theory book of ELSTRODT (1996) can supplement each of these texts.

Section 3.3 contains an analysis of decision making under uncertainty. In particular, it discusses some aspects of the objective expected utility theory. The literature on this topic is very large as well, so we again provide only a small selection of references. Since section 3.3 essentially summarizes several basic ideas portrayed in chapter 1 of EICHBERGER and HARPER (1997), their book can serve as the guiding reference. For further reference, the survey articles by KARNI and SCHMEIDLER (1991) and HAMMOND (1999), both containing long lists of additional references, may be consulted.

3.2 Modelling uncertainty

In this section, we develop a mathematical model that can capture the notions of risk and uncertainty in financial markets. We consider an economy over a fixed time interval $[0, N] \subset \mathbb{R}_+$. N is called the terminal date where we assume $N \in \mathbb{N}$. At date 0 there is uncertainty about the true state of the economy at the terminal date N. The set of possible states, however, is known. The set of all possible states ω is denoted Ω and called the state space. Subsets of Ω are called events. The family of sets that forms the set of observable events is a $\sigma-$algebra in Ω.

Definition 1 *A family $\mathcal{F}$ of sets is a $\sigma-$**algebra** in Ω if,*

1. $\Omega\in\mathcal{F}$,
2. $\mathbb{E} \in\mathcal{F} \Rightarrow\mathbb{E}^c\in\mathcal{F}$ *and*
3. $\mathbb{E}_1, \mathbb{E}_2, ... \in \mathcal{F} \Rightarrow \bigcup_{i=1}^{\infty} \mathbb{E}_i\in\mathcal{F}$.

$\mathbb{E}^c$ denotes the complement of the set $\mathbb{E}$. It is easy to see that the power set $\wp(\Omega)$ of Ω, i.e., the set of all subsets of Ω, is the largest $\sigma-$algebra in Ω and that the family $\{\emptyset, \Omega\}$ is the smallest one. On the set of observable events $\mathcal{F}$, we can define a probability measure. The probability measure carries information about the probability of observable events to occur.

Definition 2 *Let $\mathcal{F}$ be a σ-algebra in Ω. A function $\mathbf{P} : \mathcal{F} \rightarrow [0, 1]$ is a **probability measure** if,*

1. $\forall\mathbb{E} \in\mathcal{F} : \mathbf{P}(\mathbb{E}) \geq 0$,
2. $\mathbf{P}\left(\bigcup_{i=1}^{\infty} \mathbb{E}_i\right) = \sum_{i=1}^{\infty} \mathbf{P}(\mathbb{E}_i)$ *for disjoint sets* $\mathbb{E}_1, \mathbb{E}_2, ... \in \mathcal{F}$ *and*
3. $\mathbf{P}(\Omega) = 1$.

Two probability measures, $\mathbf{P}$ and $\mathbf{Q}$, defined on a σ-algebra $\mathcal{F}$ are equivalent if they agree on the same null-sets,

$$\mathbf{P}(\mathbb{E}) = 0 \Leftrightarrow \mathbf{Q}(\mathbb{E}) = 0,$$

where $\mathbb{E} \in \mathcal{F}$. A collection $(\Omega, \mathcal{F}, \mathbf{P})$ of a state space Ω, a set of observable events $\mathcal{F}$, where $\mathcal{F}$ is a $\sigma-$algebra, and a probability measure $\mathbf{P}$ defined on $\mathcal{F}$ is called a probability space.[1]

In general, securities traded in financial markets are risky bets since their future prices are uncertain. In our simple setup, a natural way to model securities with uncertain future prices is via functions of the economy's state at the terminal date. This motivates the introduction of random variables and random vectors into the model.[2]

Definition 3 *Given a probability space $(\Omega, \mathcal{F}, \mathbf{P})$, a* ***random variable*** *S is a function,*

$$S : \Omega \to \mathbb{R}_+, \omega \mapsto S(\omega),$$

that is $\mathcal{F}-$measurable, i.e., for each $\mathbb{E} \in \{[a, b[: a, b \in \mathbb{R}\}$,

$$S^{-1}(\mathbb{E}) \equiv \{\omega \in \Omega : S(\omega) \in \mathbb{E}\} \in \mathcal{F}. \tag{3.1}$$

A function,

$$S : \Omega \to \mathbb{R}_+^K, \omega \mapsto S(\omega),$$

is a ***random vector*** *if its component functions,*

$$S^k : \Omega \to \mathbb{R}_+, \omega \mapsto S^k(\omega),$$

for $k \in \{1, ..., K\}$, are $\mathcal{F}-$measurable. A random vector S is $\mathcal{F}-$measurable if all component functions S^k are $\mathcal{F}-$measurable.

It is sometimes convenient to write $S \in \mathcal{F}$ for 'S is $\mathcal{F}-$measurable' where S can be either a random variable or a random vector.

Definition 4 *Let a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ be given where Ω is finite. The* ***expectation*** *$\mathbf{E}^{\mathbf{P}}[S]$ of a random variable (or vector) S under a probability measure $\mathbf{P}$ is defined as,*

$$\mathbf{E}^{\mathbf{P}}[S] \equiv \sum_{\omega \in \Omega} \mathbf{P}(\omega) \cdot S(\omega).$$

The expectation of a random variable is real-valued whereas the expectation of a random vector is again a vector.

[1] See section 1 in chapter 1 of Bauer (1990).
[2] See section 3 in chapter 1 of Bauer (1990).

Remark 1 *With respect to definition 4, it is important to recall that we have defined random variables as taking only positive values on the real line. Otherwise we ought to be more careful.*

So far we have assumed that at date 0 there is uncertainty with regard to the state of the economy at the terminal date N. It seems more realistic, however, to assume that uncertainty resolves gradually over time. As before, let Ω be the set of all possible states of the economy at date N. Assume now that new information about the true state of the economy at date N arrives at dates $n \in \{0, 1, ..., N\}$. This concept is general enough for us to interpret the time interval $[n, n+1[$, $n < N$, between two consecutive dates as a week, a day, an hour or any other unit of 'real' time. We have[3],

Definition 5 *A **filtration** $\mathbb{F}$ is a non-decreasing family of $\sigma-$algebras in Ω, i.e., $\mathbb{F} \equiv (\mathcal{F}_n)_{n\in\{0,...,N\}}$ where $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq ... \subseteq \mathcal{F}_{N-1} \subseteq \mathcal{F}_N$.*

We call the collection $(\Omega, \mathcal{F}, \mathbb{F}, \mathbf{P})$ a filtered probability space. In the present context, the filtration is a model for the resolution of uncertainty over time. If an event $\mathbb{E} \subseteq \Omega$ is in $\mathcal{F}_n$, then it is known at date n whether the event may happen or not. In other words, if $\mathbb{E}$ is in $\mathcal{F}_n$, one can decide whether the true state ω is in $\mathbb{E}$ or not. Hence, $\mathcal{F}_n$ can be interpreted as the information set at date n. In general, we assume that $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_N = \wp(\Omega)$. Economically, this translates into 'nothing is known at the beginning of the economy' and 'everything is known at the end of the economy', respectively. The requirement that the $\mathcal{F}_n$ be non-decreasing means that information cannot be lost.

In such a dynamic context, one can generalize the idea of a random variable (vector) straightforwardly to obtain a stochastic (vector) process. This enables one to model price dynamics of securities as well.

Definition 6 *A **stochastic (vector) process** $(S_n)_{n\in\{0,...,N\}}$ is a date-ordered sequence of random variables (random vectors) $S_n, n \in \{0, ..., N\}$.*

Suppose that $(S_n)_{n\in\{0,...,N\}}$ represents the price process of a security. Since the price of a security at the terminal date depends on the state of the economy at this date, it is reasonable to assume that its price at date n depends on the information $\mathcal{F}_n$ available at date n. This gives rise to[4],

Definition 7 *A stochastic (vector) process $(S_n)_{n\in\{0,...,N\}}$ is said to be **adapted** to a filtration $\mathbb{F} =(\mathcal{F}_n)_{n\in\{0,...,N\}}$ if $\forall n : S_n$ is $\mathcal{F}_n-$measurable.*

[3] See Bauer (1990, 138).

[4] See Bauer (1990, 138).

If security price processes are adapted to the filtration, then the economy is informationally efficient.[5] In financial models, one can sometimes find the opposite situation as well: Information is generated by security price processes. To handle such situations one needs yet another concept[6],

Definition 8 *1. The $\sigma-$algebra* ***generated by a random variable (or vector)*** *S is denoted $\sigma(S)$ and is the smallest $\sigma-$algebra with respect to which S is measurable.*

2. The $\sigma-$algebra ***generated by a stochastic (vector) process,***

$$(S_n)_{n\in\{0,...,N\}},$$

up to date n is denoted,

$$\sigma(S_i : i \in \{0, ..., n\}),$$

and is the smallest $\sigma-$algebra with respect to which all random variables (vectors) $S_i, i \in \{0, ..., n\}$, are measurable.

In light of this definition, a stochastic process $(S_n)_{n\in\{0,...,N\}}$ generates the filtration $\mathbb{F} = (\mathcal{F}_n)_{n\in\{0,...,N\}}$ where $\mathcal{F}_n \equiv \sigma(S_i : i \in \{0, ..., n\})$. Of course, the stochastic process is adapted to the filtration it generates.

We now turn to martingales. Heuristically, a martingale embodies the notion of a *fair* investment. Consider a risk-neutral investor who plans to invest in a stock.[7] This investor would call the investment fair if the expected discounted price of the stock at some future date equals its present price. The investor would deny buying the stock if the actual price is higher. He would, however, always agree to buy if the price of the stock is below the expected discounted price. A stock price process satisfying the condition that the expected discounted price at any future date equals its price today is a so-called martingale.

To formally define a martingale, the concept of conditional expectation is needed. Taking expectations as proposed in definition 4 presumes that nothing is known about the state of the economy at the terminal date. In other words, the minimal $\sigma-$algebra $\{\emptyset, \Omega\}$ forms the information set. If uncertainty is gradually resolved, the information set enlarges over time and

[5] Recall the discussion about efficient markets in section 1.2. The mathematical formulation here corresponds to weak form efficiency. See also section II of FAMA (1970).

[6] See BROCK and MALLIARIS (1982, 18).

[7] An investor is *risk-neutral* if he / she is indifferent between a sure amount of money and an investment with an expected (discounted) payoff equal as high. We will give this a more precise meaning in the next section.

allows one to take better expectations. Here, *better* means that expectations are taken conditional on a relatively enlarged information set. Formally[8],

Definition 9 *Let a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbf{P})$ be given. The **conditional expectation** $\mathbf{E}_n^{\mathbf{P}}[S]$ of a random variable (vector) S given information $\mathcal{F}_n$ is the unique random variable (vector) that satisfies,*

1. *$\mathbf{E}_n^{\mathbf{P}}[S]$ is $\mathcal{F}_n-$measurable and*
2. *$\forall \mathbb{E} \in \mathcal{F}_n : \mathbf{E}^{\mathbf{P}}[\mathbf{E}_n^{\mathbf{P}}[S] \cdot \mathbf{1}_{\mathbb{E}}] = \mathbf{E}^{\mathbf{P}}[S \cdot \mathbf{1}_{\mathbb{E}}]$.*

Remark 2 *For notational simplicity, we denote the conditional expectation by $\mathbf{E}_n^{\mathbf{P}}[\cdot]$ instead of $\mathbf{E}^{\mathbf{P}}[\cdot|\mathcal{F}_n]$ as often found in the literature.*

This eventually enables the definition of a martingale[9],

Definition 10 *Let a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbf{Q})$ be given. A $\mathbb{F}$-adapted stochastic process $(S_n)_{n\in\{0,\ldots,N\}}$ is a **(vector) martingale** under the probability measure $\mathbf{Q}$ if,*

$$\forall n, s \geq 0 : \mathbf{E}_n^{\mathbf{Q}}[S_{n+s}] = S_n.$$

A probability measure $\mathbf{Q}$ that makes a stochastic process - defined on some filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbf{P})$ - a martingale is called a martingale measure. Whenever $\mathbf{Q}$ is $\mathbf{P}-$equivalent, it is called an $\mathbf{P}-$equivalent martingale measure.

It may become necessary to change from one probability measure to an equivalent probability measure, say from $\mathbf{P}$ to $\mathbf{Q}$. This is where the state price density can help.

Definition 11 *Let a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ be given where Ω is finite. For a $\mathbf{P}-$equivalent probability measure $\mathbf{Q}$, the **state price density** $\mathbf{L}$, which is a random variable, is defined by,*

$$\forall \omega \in \Omega : \mathbf{L}(\omega) \equiv \begin{cases} \frac{\mathbf{Q}(\omega)}{\mathbf{P}(\omega)} & \text{for } \mathbf{P}(\omega) \neq 0 \\ 0 & \text{for } \mathbf{P}(\omega) = 0 \end{cases}.$$

[8] See Bauer (1990, 117).

[9] See Bauer (1990, 138-139).

We conclude this section with a demonstration of how the state price density may be applied in computing expectations. Let two equivalent probability measures $\mathbf{P}$ and $\mathbf{Q}$, defined on a $\sigma-$algebra $\mathcal{F}$ in a finite state space Ω, be given. It holds that $\mathbf{E^Q}[S] = \mathbf{E^P}[\mathbf{L}S]$ for a random variable (vector) S defined on $(\Omega, \mathcal{F}, \mathbf{P})$. Easy manipulations of $\mathbf{E^Q}[S]$ verify this claim,

$$\begin{aligned}
\mathbf{E^Q}[S] &= \sum_{\omega\in\Omega} \mathbf{Q}(\omega) \cdot S(\omega) \\
&= \sum_{\omega\in\Omega} \mathbf{P}(\omega) \cdot \frac{\mathbf{Q}(\omega)}{\mathbf{P}(\omega)} \cdot S(\omega) \\
&= \sum_{\omega\in\Omega} \mathbf{P}(\omega) \cdot \mathbf{L}(\omega) \cdot S(\omega) \\
&= \mathbf{E^P}[\mathbf{L}S].
\end{aligned}$$

3.3 Decision making under uncertainty

A discussion of uncertainty in financial markets is hardly meaningful without incorporating a decision maker. We therefore take a glance at a common theory of decision making under uncertainty: the so-called *objective expected utility theory*.[10]

Decisions in financial markets are inter-temporal, meaning that actions taken by a decision maker today have consequences at some future date. In our model world, the today's purchase of a stock whose payoff at the terminal date depends on the realized state of the economy is an example for an inter-temporal decision. Since the state of the economy at the terminal date is uncertain, an investment in the stock is made under conditions of uncertainty. The plan is now to embed such decisions into the model of uncertainty developed in the previous section.

[10] In economic or financial applications, probabilities can be either objective or subjective. The former means that they are given by nature. The latter means they are assessed on the basis of individual beliefs. An economy with objective probabilities is sometimes referred to as an economy with risk, in the other case it is sometimes referred to as an economy with uncertainty.

Depending on whether there is risk or uncertainty, different theories of decision making are applicable. A common theory of decision making under risk is the *objective* expected utility theory which we briefly sketch in this section. The dominating paradigm in the other case is the *subjective* expected utility theory. For instance, Karni and Schmeidler (1991) survey both theories.

In accordance with this nomenclature, we consider in this thesis economies with *risk* only. Consequently, there is no need to distinguish between risk and uncertainty. We will therefore use the words *risk* and *uncertainty* interchangeably.

Let a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ be given and let Ω be finite. $(\Omega, \mathcal{F}, \mathbf{P})$ shall describe uncertainty in an economy that extends over the period $[0, N]$, $N \in \mathbb{N}$. Suppose that an agent may choose at date 0 from a fixed set $\mathbb{H}$ of feasible investments. The investment of the agent, i.e., the actual chosen element of $\mathbb{H}$, determines a state-contingent payoff at date N. Denote the set of all payoffs by $\mathbb{W} \subset \mathbb{R}$. Thus, a function f that maps states and investments into payoffs[11],

$$f : \Omega \times \mathbb{H} \to \mathbb{W},$$

captures the notion of an investment under uncertainty in this context. Yet this formulation is not very convenient to work with. It is more convenient to treat state-contingent payoffs directly. The set $\mathbb{W}^{\Omega}$ of all state-contingent payoffs is defined by,

$$\mathbb{W}^{\Omega} \equiv \left\{ \left(w(\omega_1), w(\omega_2), ..., w(\omega_{|\Omega|}) \right) : w(\omega_i) = f(\omega_i, h), h \in \mathbb{H} \right\}. \tag{3.2}$$

In light of (3.2), it is obvious that for an agent deciding over an investment opportunity $h \in \mathbb{H}$ is tantamount to deciding over a state-contingent payoff $w \in \mathbb{W}^{\Omega}$. To formalize this idea, let the preferences of the agent be represented by a preference relation defined on the set $\mathbb{W}^{\Omega}$. Furthermore, let the preferences of the agent be *complete*, *transitive* and *continuous*.[12] These preferences can then be represented by a utility function U of the form[13],

$$U : \mathbb{W}^{\Omega} \to \mathbb{R}. \tag{3.3}$$

In those cases where the preferences of the agent additionally satisfy a certain *state-independence* condition, (3.3) can take the particular form,

$$U : \mathbb{W}^{\Omega} \to \mathbb{R}, w \mapsto \mathbf{E}^{\mathbf{P}}\left[v(w)\right] \equiv \sum_{\omega \in \Omega} \mathbf{P}(\omega) \cdot v(w(\omega)), \tag{3.4}$$

where $v : \mathbb{W} \to \mathbb{R}$.[14] We call v the utility function of the agent and $\mathbf{E}^{\mathbf{P}}\left[v(w(\omega))\right]$ the expected utility of the agent. All agents considered in this thesis have preferences that are compatible with an expected utility representation as displayed in (3.4).

Let $w, w' \in \mathbb{W}^{\Omega}$. With the expected utility representation (3.4), the preferences of an agent translate into the following: The agent,

[11] Refer to EICHBERGER and HARPER (1997, 8).

[12] Compare VARIAN (1992, 95).

[13] MEHTA (1999) contains several proofs for the existence of utility functions.

[14] Refer to appendix B in EICHBERGER and HARPER (1997). See also section 2 of KARNI and SCHMEIDLER (1991) or sections 2-4 of HAMMOND (1999).

- prefers w over w' if,

$$\mathbf{E}^{\mathbf{P}}\left[v(w)\right] \geq \mathbf{E}^{\mathbf{P}}\left[v(w')\right],$$

- strictly prefers w over w' if,

$$\mathbf{E}^{\mathbf{P}}\left[v(w)\right] > \mathbf{E}^{\mathbf{P}}\left[v(w')\right],$$

- and is indifferent between w and w' if,

$$\mathbf{E}^{\mathbf{P}}\left[v(w)\right] = \mathbf{E}^{\mathbf{P}}\left[v(w')\right].$$

In dynamic settings, expected utility representations of preferences are possible too. For a given filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbf{P})$, (3.4) becomes,

$$U_n : \mathbb{W}^{\Omega} \to \mathbb{R}, w \mapsto \mathbf{E}_n^{\mathbf{P}}\left[v(w)\right]. \tag{3.5}$$

In (3.5), we merely replaced the expectation by the conditional expectation given the information set $\mathcal{F}_n$ at date n. Because of the assumption that the agent only cares about the state-contingent payoff at date N, this causes no trouble here. In cases in which the agent is also concerned with payoffs at dates other than N, one has to be more careful.

It is worth pointing out that agents whose preferences are compatible with an expected utility representation act dynamically consistent. Roughly speaking, this means that an expected utility maximizing agent follows his / her consumption-investment plan independent of the information gathered over time. In other words, an optimal plan is not revised by such an agent even if he receives new information.[15]

It can be observed in financial markets that investors significantly differ with respect to their appetite for risk. In the following, we address the question of how this appetite for risk can be measured.

Definition 12 *Consider an expected utility maximizing (EUM) agent with a twice continuously differentiable utility function $v : \mathbb{W} \to \mathbb{R}$ where $v'(\cdot) > 0$ over the relevant range. We say the agent is **risk-neutral** if $v''(\cdot) = 0$, **risk-averse** if $v''(\cdot) < 0$ and **risk-loving** if $v''(\cdot) > 0$ over the relevant range. $v'(\cdot)$ denotes the first derivative of $v(\cdot)$, $v''(\cdot)$ the second derivative.*

In the remainder of the thesis, we will only encounter agents that are risk-averse. Simply put, a risk-averse agent will always choose the 'sure thing'

[15] On this topic, refer to Machina (1989).

when offered the choice between a fix amount of money and an investment with an expected (discounted) payoff equally as high. In general, however, agents exhibit different degrees of risk-aversion. The degree of risk-aversion is usually measured in two different ways.[16]

Definition 13 *Consider an EUM agent with a twice continuously differentiable utility function $v : \mathbb{W} \to \mathbb{R}$ where $v'(w) > 0$. A measure for the agent's* ***absolute risk aversion*** *is,*

$$R_a \equiv -\frac{v''(w)}{v'(w)},$$

whereas a measure for his / her ***relative risk aversion*** *is given by,*

$$R_r \equiv -w \cdot \frac{v''(w)}{v'(w)}.$$

In financial applications, one often finds two particular types of decision makers. The first type is characterized by constant absolute risk aversion (CARA), the second type by constant relative risk aversion (CRRA). The following lemma, with which we conclude this section, proves useful in identifying these types of decision makers.

Lemma 14 *Consider an EUM agent with a twice continuously differentiable utility function $v : \mathbb{W} \to \mathbb{R}$ where $v'(w) > 0$.*

1. *The agent has CARA of degree $\delta > 0$ if the utility function has the functional form,*
$$v(w) = 1 - c^{-\delta w}.$$

2. *The agent has CRRA of degree $\gamma > 0$ if the utility function has the functional form,*
$$v(w) = \begin{cases} \frac{w^{1-\gamma}}{1-\gamma} & \text{for } \gamma \neq 1 \\ \ln w & \text{for } \gamma = 1 \end{cases}.$$

Proof. Applying definition 13 yields the assertions. ■

[16] See sub-section 1.4.3 of Eichberger and Harper (1997). Refer also to the seminal article of Pratt (1964).

3.4 Summary

In section 3.2, we introduced a basic model for an economy with uncertainty. The main ingredients were the state space Ω, the set of observable events $\mathcal{F}$ and the probability measure $\mathbf{P}$, which together form a probability space $(\Omega, \mathcal{F}, \mathbf{P})$. We used the concept of a random variable (random vector) to model uncertain future prices of securities. If uncertainty is gradually resolved according to a filtration $\mathbb{F}$, one works with a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbf{P})$. On such a space, one can define stochastic processes to model price dynamics of securities. A central concept in this respect is the martingale, which embodies the notion of a fair investment: The cost of the investment equals the expectation of its (discounted) payoff.

In section 3.3, we took a brief look at the objective expected utility theory of decision-making under uncertainty. We stated that under certain conditions, expected utility representations of an agent's preferences are possible. Moreover, we introduced two commonly used measures for an agent's risk aversion: absolute risk aversion and relative risk aversion.

Chapter 4

The martingale approach

4.1 Introduction

In the late 1970's, HARRISON and KREPS (1979) provided the key for the application of martingale methods to problems in finance. Martingale theory, then already a highly developed theory with many deep results, suddenly found a completely new area of application. The purpose of this chapter is to introduce this approach by building on the analysis in the previous chapter. The chapter culminates in a version of the Fundamental Theorem of Asset Pricing. The theorem relates arbitrage-freeness, i.e., the absence of opportunities to make something out of nothing, to the existence of an equivalent martingale measure. At first sight, a rather surprising result indeed.

The market model and martingale methods we explore in this chapter are mainly from section 2 of HARRISON and PLISKA (1981). However, some of the results presented date back to HARRISON and KREPS (1979). The chapter as a whole provides the tools which we will frequently apply in part III of the thesis.

The rather concise exposition in this chapter is no substitute for a more comprehensive treatment of the issues tackled within it. Aside from those already mentioned, we recommend as reference the survey article by NAIK (1995) - especially, since this article treats the application of martingale methods to portfolio choice problems. DYBVIG and ROSS (1992) present the main ideas on a rather informal level as well. The article by COX and HUANG (1989) represents a milestone in respect to applying the martingale approach to dynamic consumption problems. HE and PEARSON (1991) generalized the ideas of COX and HUANG (1989) to incomplete markets. In addition, BAXTER and RENNIE (1996), LAMBERTON and LAPEYRE (1996) or PLISKA (1997) may serve as textbook references, however, the last one is the most comprehensive regarding discrete time models.

The chapter is structured along the following lines. Section 4.2 develops the market model. Section 4.3 states central results of the martingale approach. Sections 4.4 and 4.5 contain simple examples for some of the results presented in the preceding section. While the examples in section 4.4 are more general in nature, those provided in section 4.5 purposely resemble the equilibrium models we consider in part III of the thesis. Finally, a rather comprehensive summary is found in section 4.6.

4.2 The market model

4.2.1 Primitives

We consider a model of uncertainty as examined in section 3.2. The model economy lasts for a fixed period $[0, N]$ where $N \in \mathbb{N}$ and $N < \infty$. A filtered probability space $(\Omega, \wp(\Omega), \mathbb{F}, \mathbf{P})$ is fixed where Ω is the finite state space of which each element $\omega \in \Omega$ represents one possible state of the economy at the terminal date N. New information about the true state of the economy only at date N arrives at dates $n \in \{0, 1, ..., N\}$.[1] Economic activity is also observed at these dates only. At date N, all economic activity ends. A time interval $]n, n+1[$ belongs to each date $n \leq N-1$ where there is no economic activity. The filtration $\mathbb{F} = (\mathcal{F}_n)_{n\in\{0,\ldots,N\}}$ satisfies $\mathcal{F}_0 \equiv \{\varnothing, \Omega\}$ and $\mathcal{F}_N \equiv \wp(\Omega)$. The probability measure $\mathbf{P}$ is strictly positive for all $\omega \in \Omega$, i.e., $\forall\omega \in \Omega : \mathbf{P}(\omega) > 0$. As a consequence, the probability measure $\mathbf{P}$ is uniquely defined up to equivalence.

4.2.2 Securities

There is a set $\mathbb{S}$ of $K+1$ securities available in the marketplace whose price processes are modelled by the vector process,

$$(S_n)_{n\in\{0,\ldots,N\}},$$

where $\forall n : S_n \in \mathbb{R}^{K+1}_{++}$. The first security, $k = 0$, is called *bond* and its price process is denoted,

$$(S^0_n)_{n\in\{0,\ldots,N\}}.$$

[1] Typically, models in which information only arrives at certain points in time are referred to as *discrete time models*.

The bond plays a special role since it is assumed to be risk-less.[2] Formally, risk-less means that the random variable,

$$S_n^0 : \Omega \to \mathbb{R}_{++}, \omega \mapsto S_n^0(\omega),$$

is $\mathcal{F}_{n-1}$−measurable, i.e., $\forall n \geq 1 : S_n^0 \in \mathcal{F}_{n-1}$. In other words, the actual value of S_n^0 is already known at date $n-1$. The remaining K *securities* are risky and modelled by a stochastic process each. The price process of the k−th security, $k \geq 1$, is denoted,

$$(S_n^k)_{n\in\{0,...,N\}},$$

and is adapted to the filtration $\mathbb{F}$. Recall that adapted means that the random variables,

$$S_n^k : \Omega \to \mathbb{R}_{++}, \omega \mapsto S_n^k(\omega),$$

are measurable with respect to $\mathcal{F}_n$, i.e., $\forall k, n : S_n^k(\omega) \in \mathcal{F}_n$. In other words, the actual value of S_n^k is not known until date n. Finally, we denote the *discount* process by,

$$(\beta_n)_{n\in\{0,...,N\}},$$

and define $\forall n : \beta_n \equiv (S_n^0)^{-1}$.

4.2.3 Basic definitions

We will now introduce several central expressions that are closely related to securities trading.

Definition 15 *A* ***portfolio*** *ϕ_n is a $K+1$-dimensional vector, $\phi_n \in \mathbb{R}^{K+1}$.*

A portfolio $\phi_n = (\phi_n^0, ..., \phi_n^K)$ gives the number ϕ_n^k of every security $k \in \{0, ..., K\}$ held by an agent at date n. For example, ϕ_n^0 represents the number of bonds in the portfolio ϕ_n at date n. The portfolio ϕ_0 has the natural interpretation of being the initial endowment of an agent since agents will be allowed to form a new portfolio for the first time when prices S_0 are announced. This portfolio is then labeled ϕ_1 and has to be held during the time interval $[0, 1[$.

Definition 16 *The* ***market value*** *V_n of a portfolio ϕ_n at date n is given by a function $V_n : \mathbb{R}^{K+1} \times \mathbb{R}_{++}^{K+1} \to \mathbb{R}$ where,*

$$V_n(\phi) \equiv \begin{cases} \phi_1 \cdot S_0 & \text{for } n = 0 \\ \phi_n \cdot S_n & \text{for } n \in \{1, ..., N\} \end{cases}.$$

[2] This assumption comes along with virtually no real loss of generality but it facilitates intuition considerably.

Definition 17 ϕ_n *is* ***predictable*** *if it is $\mathcal{F}_{n-1}$-measurable, i.e., if* $\forall n \geq 1 : \phi_n \in \mathcal{F}_{n-1}$.

Predictability implies that the portfolio ϕ_n is formed at $n-1$ and kept constant during the interval $[n-1, n[$. At date n, when prices S_n are announced, the portfolio has a market value of $V_n(\phi) = \phi_n \cdot S_n$. This amount can then be used, for instance, to form a new portfolio ϕ_{n+1}, which is to be held constant over the interval $[n, n+1[$, and so forth.

Definition 18 *A* ***trading strategy*** *is a predictable vector process* $(\phi_n)_{n\in\{0,...,N\}}$ *with component processes* $(\phi_n^k)_{n\in\{0,...,N\}}, k \in \{0, ..., K\}$. $(\phi_n)_{n\in\{0,...,N\}}$ *is predictable if* $\forall n \geq 1 : \phi_n$ *is predictable.*

Two other processes are directly associated with each trading strategy:

Definition 19

1. *The* ***value process*** $(V_n(\phi))_{n\in\{0,...,N\}}$ *of a trading strategy is a real-valued,* $\mathbb{F}-$*adapted process where* $V_n(\phi)$ *is given by definition 16.*

2. *The* ***gains process*** $(G_n(\phi))_{n\in\{0,...,N\}}$ *of a trading strategy is a real-valued,* $\mathbb{F}-$*adapted process where we set* $G_0 \equiv 0$ *and where* $G_n : \mathbb{R}^{K+1} \times \mathbb{R}_{++}^{K+1} \rightarrow \mathbb{R}$ *with,*

$$G_n(\phi) \equiv \sum_{i=0}^{n} \phi_i \cdot (S_i - S_{i-1}),$$

for $n \geq 1$.

In the analysis to follow, two classes of trading strategies are of particular interest: *self-financing* and *admissible* trading strategies.

Definition 20 *A trading strategy is* ***self-financing*** *if and only if*

$$(\forall n : 1 \leq n \leq N-1) : \phi_n \cdot S_n = \phi_{n+1} \cdot S_n$$

or equivalently, if and only if,

$$(\forall n : 1 \leq n \leq N-1) : V_n(\phi) = V_0(\phi) + G_n(\phi).$$

Neither are funds withdrawn nor additional funds invested at dates between $n = 1$ *and* $n = N-1$.

Definition 21 *A trading strategy is* ***admissible*** *if it is self-financing and if its value process* $(V_n(\phi))_{n\in\{0,\ldots,N\}}$ *satisfies* $\forall n : V_n(\phi) \geq 0$. $\mathbb{T}$ *denotes the set of all admissible trading strategies.*

Agents who can only implement admissible trading strategies are not allowed to produce a position of debt. In other words, agents cannot implement trading strategies that possibly lead to bankruptcy. Moreover, this implies that they must have non-negative initial endowments.

To conclude this sub-section, assume that markets are *perfect* and *perfectly liquid.*[3]

4.2.4 Agents

The population of the economy consists of a set $\mathbb{I}$ of many small agents. It can either $\mathbb{I} \subseteq \mathbb{R}_+$ or $\mathbb{I} \subseteq \mathbb{N}$ hold. Agents' preferences are respectively given by a preference relation that is strictly increasing, meaning that agents prefer more to less. The preference relation of each agent $i \in \mathbb{I}$ is defined over the set of state-contingent consumption payoffs in the space $\mathbb{R}_+^{|\Omega|}$. Each agent has strictly positive initial wealth at date 0. The set of feasible state-contingent consumption payoffs, given the initial wealth of agent $i \in \mathbb{I}$, is denoted $\mathbb{B}_i$. We assume that $\mathbb{B}_i \subseteq \mathbb{R}_+^{|\Omega|}$ is closed and convex. Moreover, we assume that the preferences of the agents allow for expected utility representations as in (3.4). In summary, the problem of agent $i \in \mathbb{I}$ is given by,

$$\max_{w\in\mathbb{B}_i} \mathbf{E}_0^{\mathbf{P}}[v_i(w)],$$

where $v_i : \mathbb{R}_+ \to \mathbb{R}$ is the utility function of agent i.[4] There is *perfect competition* (or *price taking*) among agents, as well as *complete* and *symmetric information.*

This completes the description of the market model.

[3] See section 1.1 on these model assumptions.

[4] For the results presented in this chapter, we could allow the set $\mathbb{I}$ of agents to include more general types of agents. Refer to Naik (1995).

Summary

In summary, one ends up with,

Definition 22 *The* ***market model*** $\mathcal{M}$ *is a collection of,*

- *a finite state space* Ω,
- *a filtration* $\mathbb{F}$,
- *a strictly positive probability measure* $\mathbf{P}$ *defined on* $\wp(\Omega)$,
- *a terminal date* $N \in \mathbb{N}$, $N < \infty$,
- *a set,*
$$\mathbb{S} \equiv \{(S_n^k)_{n \in \{0,...,N\}} : k \in \{0, ..., K\}\},$$
of $K + 1$ *strictly positive security price processes and,*
- *a set* $\mathbb{I}$ *of (small) expected utility maximizing agents.*

We write $\mathcal{M} = \{(\Omega, \wp(\Omega), \mathbb{F}, \mathbf{P}), N, \mathbb{S}, \mathbb{I}\}$.

4.3 Central results

This section's main objective is to state the Fundamental Theorem of Asset Pricing. In economic terms, central topics of this section are *arbitrage-freeness*, *arbitrage-free contingent claim prices* and *market completeness.*

A central problem in financial economics is the determination of fair contingent claim prices. As noted earlier, one can think of contingent claims as being derivative securities, consumption payoffs or arbitrary claims payable at N. In order to proceed, however, a formal definition of a contingent claim is needed.

Definition 23 *A* ***contingent claim*** $A_N \in \mathbb{R}_+^{|\Omega|}$ *is a non-negative random variable,*
$$A_N : \Omega \to \mathbb{R}_+, \omega \mapsto A_N(\omega).$$
$A_N(\omega)$ *is the amount payable if state* $\omega \in \Omega$ *unfolds.*

A natural question that arises is that of the attainability of contingent claims.

Definition 24 *A contingent claim A_N is* ***attainable*** *if there exists an admissible trading strategy that generates its payoff at maturity, $V_N(\phi) = A_N$, and if $A_0 \equiv V_0(\phi)$ is the price or value of the contingent claim at $n = 0$. $\mathbb{A} \subseteq \mathbb{R}_+^{|\Omega|}$ denotes the set of attainable contingent claims.*

Another question is which contingent claims are super-replicable.

Definition 25 *A contingent claim A_N is* ***super-replicable*** *if there exists an admissible trading strategy that generates a payoff dominating the contingent claim's payoff, $V_N(\phi) \geq A_N$, and if $A_0 \equiv V_0(\phi)$ are the associated* ***super-replication costs*** *at $n = 0$.*[5] *Such a trading strategy is said to super-replicate the contingent claim. $\mathbb{A}^* \subseteq \mathbb{R}_+^{|\Omega|}$ denotes the set of super-replicable contingent claims.*

Obviously, the set of attainable contingent claims $\mathbb{A}$ is in general a sub-set of the set of super-replicable contingent claims $\mathbb{A}^*$.

Definition 26 *A* ***linear price system*** *is a positive linear function $\tau : \mathbb{A} \rightarrow \mathbb{R}_+$ with,*

$$\begin{array}{l} \forall a, b \in \mathbb{R}_+, \\ \forall A_N, A'_N \in \mathbb{A} \end{array} : \left\{ \begin{array}{l} \tau(A_N) = 0 \Leftrightarrow A_N = 0 \\ \tau(a \cdot A_N + b \cdot A'_N) = a \cdot \tau(A_N) + b \cdot \tau(A'_N) \end{array} \right. .$$

$\mathbb{P}$ denotes the set of all price systems that are consistent with the market model $\mathcal{M}$, i.e., where,

$$(\forall \tau \in \mathbb{P} \text{ and } \forall \phi \in \mathbb{T}) : \tau[V_N(\phi)] = V_0(\phi).$$

To further analyze pricing issues, the formal concept of an arbitrage opportunity proves useful.

Definition 27 *An* ***arbitrage opportunity*** *is a self-financing trading strategy whose value process satisfies $V_0(\phi) = 0$ and $V_N(\phi) \geq 0$ with $\mathbf{E}_0^{\mathbf{P}}[V_N(\phi)] > 0$.*

It should be clear that a security market where arbitrage opportunities exist cannot be in equilibrium. An arbitrage opportunity arises, for example, if there are two or more different prices for the same contingent claim. A simple arbitrage strategy would then be to sell the contingent claim at a high price and to buy it at a lower price, thereby locking in the difference

[5] Sometimes the definition includes the requirement that the trading strategy be chosen such that it minimizes the super-replication costs A_0.

as a risk-less profit. The profit is risk-less because the payoffs at date N of one contingent claim long and one contingent claim short perfectly compensate each other. Of course, every agent would try to achieve such a risk-less profit.[6] Since agents' budget sets are unbounded in the presence of arbitrage opportunities, markets would inevitably be in disequilibrium. That is why the absence of arbitrage opportunities is a crucial property of equilibrium models. However, from an economic point of view, the assumption of arbitrage-freeness is rather mild.[7]

In light of the above considerations, establishing conditions that guarantee the absence of arbitrage opportunities in the market model $\mathcal{M}$ is obviously of great importance, which is what we will do next. To begin with, denote $\mathbb{Q}$ to be the set of all probability measures $\mathbf{Q}$ that are equivalent to $\mathbf{P}$ and that make the discounted security price process $(\beta_n S_n)_{n\in\{0,\ldots,N\}}$ a martingale. At this point, the main concepts for reproducing some of the central results of the martingale approach are complete.

Proposition 28 (Harrison and Pliska (1981, prop. 2.6)) *There is a one-to-one correspondence in the market model* $\mathcal{M} = \{(\Omega, \wp(\Omega), \mathbb{F}, \mathbf{P}), N, \mathbb{S}, \mathbb{I}\}$ *between price systems* $\tau \in \mathbb{P}$ *and* $\mathbf{P}-$*equivalent martingale measures* $\mathbf{Q} \in \mathbb{Q}$ *via,*

1. $\tau(A_N) = \mathbf{E}_0^{\mathbf{Q}}[\beta_N \cdot A_N]$ *and*
2. $\mathbf{Q}(\mathbb{E}) = \tau\left(S_N^0 \mathbf{1}_{\mathbb{E}}\right), \mathbb{E} \in \wp(\Omega)$.

Proof. Harrison and Pliska (1981, 227). ∎

Proposition 28 states that there is a one-to-one correspondence between a completely economic concept, a price system, and a completely probabilistic concept, a martingale measure. It should be clear that this has important implications for the market model. The importance is impressively illustrated by the following theorem.

Theorem 29 (Fundamental Theorem of Asset Pricing) *Consider the market model* $\mathcal{M} = \{(\Omega, \wp(\Omega), \mathbb{F}, \mathbf{P}), N, \mathbb{S}, \mathbb{I}\}$. *The following four statements are equivalent:*

1. *There are no arbitrage opportunities in the market model* $\mathcal{M}$.
2. *The set* $\mathbb{Q}$ *of* $\mathbf{P}-$*equivalent martingale measures is non-empty.*

[6] *Local non-satiation,* as defined in Varian (1992, 96), is a sufficient condition.

[7] For a discussion on this and other possible model assumptions (e.g., the law of one price) refer to section 1.2 of Pliska (1997).

3. *The set $\mathbb{P}$ of consistent linear price systems is non-empty.*

4. *The expected utility maximization problem of agent $i \in \mathbb{I}$ has a solution.*

Proof. The equivalence between 1. and 2. is proven, for instance, in SCHACHERMAYER (1992). Different approaches to prove this equivalence are reviewed in DELBAEN (1999). DELBAEN and SCHACHERMAYER (1997) summarize several results regarding this particular equivalence. The article by HARRISON and PLISKA (1981, 228-229) contains a proof of the equivalence between 1., 2. and 3. Finally, NAIK (1995, 36-39) proves the equivalence between all four statements. The proof of NAIK (1995) even includes more general types of agents. ■

Remark 3 *The expression 'Fundamental Theorem of Asset Pricing' was originally proposed by* DYBVIG *and* ROSS (1992) *in an earlier edition of the New Palgrave Dictionary of Money and Finance.*

In part III, Theorem 29 will prove very powerful. Starting with the economically plausible assumption that a market model is free of arbitrage opportunities, Theorem 29 guarantees the existence of a solution to the optimization problem of an expected utility maximizing agent. It also implies that there is an equivalent martingale measure. Why this last implication is so important should become clear in light of the following two results.

Corollary 30 (HARRISON **and** PLISKA **(1981, 228))** *If the market model $\mathcal{M} = \{(\Omega, \wp(\Omega), \mathbb{F}, \mathbf{P}), N, \mathbb{S}, \mathbb{I}\}$ is arbitrage-free, then there exists a unique price A_0 associated with any contingent claim $A_N \in \mathbb{A}$. It satisfies $\forall \mathbf{Q} \in \mathbb{Q}$: $A_0 = \mathbf{E}_0^{\mathbf{Q}}[\beta_N \cdot A_N]$.*

For arbitrary dates $n \in \{0, ..., N\}$, the following result emerges.

Proposition 31 (HARRISON **and** PLISKA **(1981, proposition 2.9.))** *For every $A_N \in \mathbb{A}$,*

$$\beta_n \cdot V_n(\phi) = \mathbf{E}_n^{\mathbf{Q}}[\beta_N \cdot A_N],$$

for all dates $n \in \{0, ..., N\}$, for all trading strategies $(\phi_n)_{n\in\{0,...,N\}} \in \mathbb{T}$ that generate A_N and for all $\mathbf{P}-$equivalent martingale measures $\mathbf{Q} \in \mathbb{Q}$.

Proof. HARRISON and PLISKA (1981, 230). ■

Suppose Theorem 29 applies to the market model $\mathcal{M}$. From corollary 30 and proposition 31, one obtains as the date n price A_n of an attainable contingent claim A_N,

$$A_n = \beta_n^{-1} \cdot \mathbf{E}_n^{\mathbf{Q}}[\beta_N \cdot A_N], \tag{4.1}$$

with everything defined as before and particularly, $\mathbf{Q} \in \mathbb{Q}$. Equation (4.1) states that the date n price of an attainable contingent claim is simply the expectation of its discounted payoff under an appropriate probability measure multiplied by the price of the bond.[8] This seems remarkably simple. Yet considerable problems usually arise when one wishes to apply this method to the real marketplace, i.e., when a specific price has to be computed.

In applications, it is sometimes helpful to have yet another concept available, namely the martingale basis.

Definition 32 *A **martingale basis** $\mathbb{Q}^B \equiv \{\mathbf{Q}_1^B, ..., \mathbf{Q}_J^B\}$ of $\mathbb{Q}$ is a finite set of J probability measures defined on $(\Omega, \wp(\Omega))$ such that each $\mathbf{Q} \in \mathbb{Q}$ can be expressed as a linear combination of the $\mathbf{Q}_j^B$.*

Remark 4

1. *Notice that the $\mathbf{Q}_j^B$ do not necessarily have to be $\mathbf{P}-$equivalent.*
2. *The existence of a martingale basis follows from Theorem 3.4.6 in* Schneider *and* Barker (1973) *after having observed that $\mathbb{Q} \subseteq \mathbb{R}_{++}^{|\Omega|}$ where $|\Omega| < \infty$ by assumption.*

The advantage of having this concept available becomes obvious with the following useful result.

Lemma 33 *If $\mathbb{Q}^B$ is a martingale basis of $\mathbb{Q}$, then the following equivalence holds:*

$$\begin{aligned} &\forall \mathbf{Q} \in \mathbb{Q} : A_0 = \mathbf{E}_0^{\mathbf{Q}} \left[\beta_N \cdot A_N\right] \\ \Leftrightarrow\ &\forall \mathbf{Q}_j^B \in \mathbb{Q}^B : A_0 = \mathbf{E}_0^{\mathbf{Q}_j^B} \left[\beta_N \cdot A_N\right]. \end{aligned}$$

Proof. The lemma follows from standard results of linear algebra.[9] ■

A brief discussion of market completeness should conclude this sub-section.

Definition 34 *The market model $\mathcal{M} = \{(\Omega, \wp(\Omega), \mathbb{F}, \mathbf{P}), N, \mathbb{S}, \mathbb{I}\}$ is **complete** if it is arbitrage-free and if every contingent claim is attainable, or equivalently, if $\mathbb{A} = \mathbb{R}_+^{|\Omega|}$.*

In discrete time, a convenient characterization of complete markets is possible.

---

[8] Note that $\beta_n^{-1} \equiv S_n^0$.

[9] For a brief discussion of this lemma and an illustration of its application to optimal consumption problems refer to Pliska (1997, 59).

Proposition 35 (HARRISON **and** KREPS **(1979)**) *Suppose that the market model* $\mathcal{M} = \{(\Omega, \wp(\Omega), \mathbb{F}, \mathbf{P}), N, \mathbb{S}, \mathbb{I}\}$ *is arbitrage-free. The market model* $\mathcal{M}$ *is complete if and only if* $\mathbb{Q}$ *is a singleton.*

Proof. HARRISON and KREPS (1979) do not give a formal proof but the argument is straightforward. In discrete time, the resolution of uncertainty can generally be represented by so-called event trees.[10] If one calculates martingale branch probabilities, one observes that these are unique if markets are complete. The corresponding equivalent martingale measure is then unique as well. Hence, $\mathbb{Q}$ is a singleton if markets are complete.

The converse statement follows from the observation that if markets are incomplete then there are always many probability measures contained in $\mathbb{Q}$. In fact, there are an infinite number of such probability measures in general. So $\mathbb{Q}$ has to be a singleton for markets to be complete.

For a formal proof refer, for example, to LAMBERTON and LAPEYRE (1996, 9-10). ■

Remark 5 *As an aside, we want to demonstrate that, under certain circumstances, one can interpret martingale probabilities as* ARROW *security prices.*[11] *The defining property of an* ARROW *security is that it pays off one unit in a predetermined state and nothing in other states. To make our argument, assume for the moment that the market model is complete and that interest rates are zero. Consider an arbitrary* ARROW *security, say, for example, the one that pays in state* $\widetilde{\omega} \in \Omega$. *Given the unique* $\mathbf{P}-$*equivalent martingale measure* $\mathbf{Q}$, *its price* $A_0^{\widetilde{\omega}}$ *at date* 0, *according to proposition 30 must be,*

$$\begin{aligned} A_0^{\widetilde{\omega}} &= \mathbf{E}_0^{\mathbf{Q}}[(0, ..., \underbrace{1}_{\widetilde{\omega}-th\ element}, ..., 0)] \\ &= \mathbf{Q}(\widetilde{\omega}) \cdot 1. \end{aligned}$$

Consequently, for there to be no arbitrage, the price of the chosen ARROW *security must equal the probability under the unique* $\mathbf{P}-$*equivalent martingale measure for state* $\widetilde{\omega}$ *to pertain.*

[10] Event trees are one possible way to graphically represent filtrations. The main feature of these trees is that every node has a unique predecessor. They should be carefully distinguished from recombining trees that are sometimes used to illustrate the evolution of the stock price process in the binomial option pricing model. In recombining trees, nodes may have more than one predecessor. We will return to this topic in chapter 5 in the context of the COX, ROSS, and RUBINSTEIN (1979) model.

[11] Yet another expression for ARROW security price is *state price.* The premier appearence of ARROW securities was in ARROW (1964).

The examples contained in the two subsequent sections will illustrate some of the notions presented in this section.

4.4 Two date examples

This section provides simple examples for two date economies that are intended to illustrate the application of the methods and results as portrayed in the previous section. The examples include both complete and incomplete markets settings. Although the examples may seem rather simplistic, the basic ideas behind the martingale approach and the main differences in applying it to either complete or incomplete markets settings should become apparent. The extension to more general settings is straightforward.

All examples in this section are based on an extremely simplified market model $\mathcal{M}_0 = \{(\Omega, \wp(\Omega), \mathbb{F}, \mathbf{P}), N = 1, \mathbb{S}^1, \mathbb{I}\}$. $\mathbb{S}^1$ indicates that there is only one risky security, in the sequel called *stock*. Furthermore, in $\mathcal{M}_0$ the risk-less interest rate is set equal to zero.

4.4.1 Option pricing in complete markets

Consider the market model $\mathcal{M}_0$ with a model economy that lasts for the period $[0, 1]$. Two states of the world are possible at $n = 1$, $\Omega = \{u, d\}$, both of which occur with strictly positive probability, $\mathbf{P}(u), \mathbf{P}(d) > 0$. The following securities payoff structure at $n = 1$ is exogenously given,

	bond price S_1^0	stock price S_1^1
state u	1	12
state d	1	8

.

Assume that the stock trades for 10 at $n = 0$. With this, the unique equivalent martingale measure for the stock is easily computed to satisfy $\mathbf{Q}(u) = \mathbf{Q}(d) = 0.5$. We now introduce a call option into the economy, which is defined by,

$$C_1 : \mathbb{R}_{++} \to \mathbb{R}_+, S_1^1 \mapsto \max\{S_1^1 - 11, 0\}.$$

The option has a payoff of 1 if state u unfolds and a payoff of 0 if state d unfolds. Since the $\mathbf{P}-$equivalent martingale measure is unique, the market model $\mathcal{M}_0$ is complete with the implication that the option can be replicated by a portfolio consisting of shares of the stock and units of the bond. Formally, $\mathbb{A} = \mathbb{R}_+^2$. We will demonstrate two different ways of deriving the arbitrage-free price of the option, namely,

- determining the hedge costs of the option and

- taking expectations of the option's state-contingent payoff under the unique $\mathbf{P}-$equivalent martingale measure.

On the one hand, one can determine the hedge portfolio by considering the linear system,

$$\phi_1^0 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \phi_1^1 \cdot \begin{pmatrix} 12 \\ 8 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$\Rightarrow \begin{cases} \phi_1^0 + 12 \cdot \phi_1^1 = 1 \\ \phi_1^0 + 8 \cdot \phi_1^1 = 0 \end{cases},$$

whose unique solution is $(\phi_1^0, \phi_1^1) = (-2, 0.25)$. ϕ_1^0 denotes the number of units of the bond and ϕ_1^1 the number of shares of the stock contained in the hedge portfolio. It can be easily verified that the portfolio (ϕ_1^0, ϕ_1^1) has the same payoff as the option. Correspondingly, the hedge costs are,

$$C_0 = -2 \cdot 1 + 0.25 \cdot 10 = 0.5.$$

Alternatively stated, the absence of arbitrage, as implied by the existence of an $\mathbf{P}-$equivalent martingale measure, enforces a price $C_0 = 0.5$ for the call option.[12]

On the other hand, corollary 30 asserts that the price of the option equals the expectation of its payoff under any equivalent martingale measure. And indeed, taking expectations yields the same price for the call option,

$$\begin{aligned} C_0 &= \mathbf{E}_0^{\mathbf{Q}}[C_1] \\ &= \mathbf{E}_0^{\mathbf{Q}}[\max\{S_1^1 - 11, 0\}] \\ &= 0.5 \cdot 1 + 0.5 \cdot 0 \\ &= 0.5, \end{aligned} \tag{4.2}$$

as desired.

Clearly, there is a close connection between both ways of pricing the option. Just notice that the payoff at date $n = 1$ of the hedge portfolio equals the payoff of the option,

$$\begin{aligned} C_1 &\overset{!}{=} \phi_1^0 + \phi_1^1 \cdot S_1^1 \\ &= -2 + 0.25 \cdot S_1^1. \end{aligned} \tag{4.3}$$

[12] The procedure applied here is known as *two state option pricing* and was originally developed in SHARPE (1978). It is a rather special case of the binomial option pricing model of COX, ROSS, and RUBINSTEIN (1979). See also the informal discussion in section 1.1.

Substituting for the payoff C_1 in (4.2) yields,

$$\begin{aligned} C_0 &= \mathbf{E}_0^{\mathbf{Q}}[\phi_1^0 + \phi_1^1 \cdot S_1^1] \\ &= \phi_1^0 + \phi_1^1 \cdot \mathbf{E}_0^{\mathbf{Q}}[S_1^1] \qquad (4.4) \\ &= \phi_1^0 + \phi_1^1 \cdot S_0^1 \qquad (4.5) \\ &= -2 + 0.25 \cdot 10 \\ &= 0.5. \end{aligned}$$

(4.4) follows from the linearity of the expectation while (4.5) follows from the very definition of the martingale measure.

Remark 6 *The call option happens to be an* Arrow *security in the example economy since it pays one unit in state u and nothing in state d. Its price of* 0.5 *coincides with the martingale probability for state u which is what we have already pointed out in remark 5 for a more general case.*

4.4.2 Option pricing in incomplete markets

This example is essentially an extension of the previous one. Contrary to the previous setting, there are now three different states possible. It now holds $\Omega = \{u, m, d\}$ in the market model $\mathcal{M}_0$. Assume $\mathbf{P}(u), \mathbf{P}(m), \mathbf{P}(d) > 0$. Economically, this difference has, as we will see, far-reaching implications provided the set of securities remains unchanged, as it does in our case. The new payoff structure of the two available securities is,

	bond price S_1^0	stock price S_1^1
state u	1	12
state m	1	10
state d	1	8

In this case, the identification of the $\mathbf{P}-$equivalent martingale measure(s) is not as simple as before but almost equally straightforward. Recalling the definition of a martingale, we look for a $\mathbf{P}-$equivalent probability measure $\mathbf{Q}$ such that,

$$\begin{aligned} & S_0^1 \stackrel{!}{=} \mathbf{E}_0^{\mathbf{Q}}[S_1^1] \\ \Rightarrow \quad & 10 \stackrel{!}{=} q_u \cdot 12 + q_m \cdot 10 + q_d \cdot 8. \end{aligned}$$

Fortunately, since we are looking for $\mathbf{P}-$equivalent probability measures, we have some more information about the probabilities $q_\omega, \omega \in \Omega$. Altogether,

$$\begin{cases} q_u \cdot 12 + q_m \cdot 10 + q_d \cdot 8 = 10 \\ q_u + q_m + q_d = 1 \\ q_u, q_m, q_d > 0 \end{cases}.$$

Straightforward calculations yield the solutions,

$$\begin{cases} q_m = 1 - 2 \cdot q_u \\ q_d = q_u \\ q_u \in \left]0, \frac{1}{2}\right[\end{cases},$$

or equivalently,

$$\mathbb{Q} = \left\{ \mathbf{Q} \in \mathbb{R}^3_{++} : \mathbf{Q} = (\rho, 1 - 2 \cdot \rho, \rho) \text{ and } \rho \in \left]0, \frac{1}{2}\right[\right\}.$$

Obviously, there are over-countably many probability measures under which the stock price process becomes a martingale. As a consequence, the market model $\mathcal{M}_0$ is now incomplete according to proposition 35. The set of attainable contingent claims is,

$$\mathbb{A} = \left\{ A_1 \in \mathbb{R}^3_+ : A_1 = \nu \cdot \begin{pmatrix} 12 \\ 10 \\ 8 \end{pmatrix} + \nu' \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} ; \nu, \nu' \in \mathbb{R} \right\},$$

Because of market incompleteness, $\mathbb{A}$ is a proper subset of $\mathbb{R}^3_+$.

Next we want to construct a martingale basis for the purpose of reducing the number of martingale measures one has to take care of. Candidate probability measures that suggest themselves are those that correspond to $\rho = 0$ and $\rho = \frac{1}{2}$, the infimum and the supremum of the range of possible ρ values, respectively. In fact,

$$\begin{aligned} \mathbf{Q}_1^B &= (0, 1, 0) \text{ and} \\ \mathbf{Q}_2^B &= \left(\frac{1}{2}, 0, \frac{1}{2} \right), \end{aligned}$$

constitute a martingale basis $\mathbb{Q}^B$. Given an arbitrary ρ, the defining property of a martingale basis can be verified by,

$$(1 - \theta) \cdot \mathbf{Q}_1^B + \theta \cdot \mathbf{Q}_2^B = \mathbf{Q}(\rho). \tag{4.6}$$

For any $\rho \in]0, \frac{1}{2}[$, (4.6) yields a unique θ. In particular, $\theta = 2 \cdot \rho$.

We are now in a position where we can conveniently price attainable contingent claims. A call option defined by,

$$C_1 : \mathbb{R}_{++} \to \mathbb{R}_+, S_1^1 \mapsto \max\{S_1^1 - 5, 0\},$$

is attainable via the hedge portfolio $(\phi_1^0, \phi_1^1) = (-5, 1)$. Given a $n = 0$ stock price of 10, simple calculations deliver a price of $C_0 = 5$ for the call option.

By pricing the option the martingale way, one obtains,

$$\begin{aligned} C_0 &= \mathbf{E}_0^{\mathbf{Q}_1^B}[C_1] \\ &= 0 \cdot 7 + 1 \cdot 5 + 0 \cdot 3 \\ &= 5, \end{aligned}$$

and,

$$\begin{aligned} C_0 &= \mathbf{E}_0^{\mathbf{Q}_2^B}[C_1] \\ &= \frac{1}{2} \cdot 7 + 0 \cdot 5 + \frac{1}{2} \cdot 3 \\ &= 5. \end{aligned}$$

By lemma 33 and corollary 30 we can verify that $C_0 = 5$ is indeed the right price for the option. Option prices obtained by the hedge argument and by the martingale argument are the same, which should not cause any surprise by now.

4.4.3 Optimal consumption in complete markets

For this example, consider an expected utility maximizing agent who lives in the economy as outlined in sub-section 4.4.1, i.e., where the market model $\mathcal{M}_0$ is complete. However, we additionally assume that $\mathbf{P}(u) = \frac{1}{3}$ and $\mathbf{P}(d) = \frac{2}{3}$ holds. The agent derives utility from consumption only at date $n = 1$. The agent is endowed with a utility function of the form,

$$\begin{aligned} v : \mathbb{R}_{++} \to \mathbb{R}, w \mapsto v(w) \\ \text{with } v(w) = \ln w. \end{aligned}$$

w denotes actual consumption at date $n = 1$. The logarithmic utility function implies CRRA of 1. At $n = 0$, the agent maximizes his / her expected utility over $n = 1$ consumption. The initial wealth is $W_0 = 10$. Accordingly, the whole problem is to,

$$\max_{W_1 \in \mathbb{R}^2_{++}} \mathbf{E}_0^{\mathbf{P}}[\ln W_1] \tag{4.7}$$

$$\text{s.t. } \mathbf{E}_0^{\mathbf{Q}}[W_1] = W_0. \tag{4.8}$$

Because of the strict monotonicity of the utility function $v(\cdot)$, the budget constraint (4.8) is binding. It states that the agent can choose among those consumption payoffs W_1 that cost W_0. With the respective parameter specifications, one obtains from problem (4.7) and (4.8),

$$\begin{aligned}\max_{W_1(u),W_1(d)} \ & \frac{1}{3}\cdot\ln W_1(u)+\frac{2}{3}\cdot\ln W_1(d)\\ \text{s.t. } & \frac{1}{2}\cdot W_1(u)+\frac{1}{2}\cdot W_1(d)=10.\end{aligned}$$

As the LAGRANGIAN function[13] emerges,

$$\begin{aligned}\mathcal{L}(W_1(u),W_1(d),\lambda) \quad = \quad & \frac{1}{3}\cdot\ln W_1(u)+\frac{2}{3}\cdot\ln W_1(d)\\ & +\lambda\cdot\left(10-\frac{1}{2}\cdot W_1(u)-\frac{1}{2}\cdot W_1(d)\right).\end{aligned}$$

The first order conditions, which are both necessary and sufficient in this special case, are,

$$\begin{cases}\frac{\partial\mathcal{L}}{\partial W_1(u)}=\frac{1}{3\cdot W_1(u)}-\frac{1}{2}\cdot\lambda\stackrel{!}{=}0\\ \frac{\partial\mathcal{L}}{\partial W_1(d)}=\frac{2}{3\cdot W_1(d)}-\frac{1}{2}\cdot\lambda\stackrel{!}{=}0\\ \frac{\partial\mathcal{L}}{\partial\lambda}=10-\frac{1}{2}\cdot W_1(u)-\frac{1}{2}\cdot W_1(d)\stackrel{!}{=}0\end{cases}\tag{4.9}$$

From the first two,

$$W_1(d)=2\cdot W_1(u).$$

Substituting for $W_1(d)$ in the budget constraint eventually yields $W_1 = (W_1(u), W_1(d)) = \left(\frac{20}{3},\frac{40}{3}\right)$ as the optimal solution. The agent has to buy $\frac{80}{3}$ units of the bond and to sell short $\frac{5}{3}$ shares of the stock to produce the optimal payoff, $(\phi_1^0,\phi_1^1)=\left(\frac{80}{3},-\frac{5}{3}\right)$. The optimal objective value is approximately 2.36.

4.4.4 Optimal consumption in incomplete markets

In this example, we move the agent of the previous example into the economy described in sub-section 4.4.2. As shown, the market model $\mathcal{M}_0$ is incomplete there. Assume that $\mathbf{P}(u)=\mathbf{P}(m)=\mathbf{P}(d)=\frac{1}{3}$. The problem of the agent in this setting is[14],

$$\max_{W_1\in\mathbb{A}} \mathbf{E}_0^{\mathbf{P}}[\ln W_1]\tag{4.10}$$

[13] For a version of the Theorem of LAGRANGE refer to SCHINDLER (1997, 203-212).

[14] Referring to a discrete time, discrete space setting with incomplete markets, HE and PEARSON (1991) point out: "In that setting, the set of feasible consumption bundles can be defined by budget constraints formed using the extreme points of the closure of the set of Arrow-Debreu state prices consistent with no arbitrage ..." Combining this insight with remark 5 explains our particular choice for the martingale basis.

$$\text{s.t. } \mathbf{E}_0^{\mathbf{Q}_1^B}[W_1] = W_0 \tag{4.11}$$
$$\mathbf{E}_0^{\mathbf{Q}_2^B}[W_1] = W_0. \tag{4.12}$$

The attainable consumption payoffs W_1 among those the agent is allowed to choose now have to satisfy two constraints. If they do so, lemma 33 ensures that W_1 satisfies $\forall \mathbf{Q} \in \mathbb{Q} : \mathbf{E}_0^{\mathbf{Q}}[W_1] = W_0$. Note that because of $v(\cdot)$ being strictly monotonic, both constraints (4.11) and (4.12) are binding. The parameterized problem is,

$$\begin{aligned} \max_{W_1(u),W_1(m),W_1(d)} & \frac{1}{3} \cdot (\ln W_1(u) + \ln W_1(m) + \ln W_1(d)) \\ \text{s.t. } & 0 \cdot W_1(u) + 1 \cdot W_1(m) + 0 \cdot W_1(d) = 10 \\ & \frac{1}{2} \cdot W_1(u) + 0 \cdot W_1(m) + \frac{1}{2} \cdot W_1(d) = 10, \end{aligned}$$

giving rise to the LAGRANGIAN function,

$$\begin{aligned} \mathcal{L}(W_1(u), W_1(m), W_1(d), \lambda_1, \lambda_2) &= \frac{1}{3} \cdot (\ln W_1(u) + \ln W_1(m) + \ln W_1(d)) \\ &\quad + \lambda_1 \cdot (10 - W_1(m)) \\ &\quad + \lambda_2 \cdot \left(10 - \frac{1}{2} \cdot W_1(u) - \frac{1}{2} \cdot W_1(d)\right). \end{aligned}$$

The first order conditions are,

$$\left\{ \begin{array}{l} \frac{\partial \mathcal{L}}{\partial W_1(u)} = \frac{1}{3 \cdot W_1(u)} - \frac{1}{2} \cdot \lambda_2 \overset{!}{=} 0 \\ \frac{\partial \mathcal{L}}{\partial W_1(m)} = \frac{1}{3 \cdot W_1(m)} - \lambda_1 \overset{!}{=} 0 \\ \frac{\partial \mathcal{L}}{\partial W_1(d)} = \frac{1}{3 \cdot W_1(d)} - \frac{1}{2} \cdot \lambda_2 \overset{!}{=} 0 \\ \frac{\partial \mathcal{L}}{\partial \lambda_1} = 10 - W_1(m) \overset{!}{=} 0 \\ \frac{\partial \mathcal{L}}{\partial \lambda_2} = 10 - \frac{1}{2} \cdot W_1(u) - \frac{1}{2} \cdot W_1(d) \overset{!}{=} 0 \end{array} \right. .$$

From these, one can derive the unique optimal solution as being $W_1 = (W_1(u), W_1(m), W_1(d)) = (10, 10, 10)$. The agent seeks complete insurance in the sense that he / she achieves a state-independent payoff at $n = 1$. The agent has to invest all the initial wealth in the bond, $(\phi_1^0, \phi_1^1) = (10, 0)$, to accomplish the desired payoff. The optimal objective value is then approximately 2.3.

4.5 Three date examples

The benefits of applying martingale methods become completely apparent in dynamic settings only. However, contingent claim pricing is formally the same in both static and dynamic settings: The price of the contingent claim equals the expectation of its (discounted) payoff under all equivalent martingale measures. An investor does not have to be concerned with the number of periods between today and the maturity date of the contingent claim. His only task is to compute the expected value(s).

However, since we will draw on martingale techniques several times throughout part III when analyzing optimal consumption problems (or portfolio choice problems), we will illustrate this using two examples for a dynamic setting. The two examples differ in respect to market completeness of the market model $\mathcal{M}_1 = \{(\Omega, \wp(\Omega), \mathbb{F}, \mathbf{P}), N = 2, \mathbb{S}^1, \mathbb{I}\}$ on which the examples are based.

4.5.1 Optimal consumption in complete markets

Consider the market model $\mathcal{M}_1$ and an expected utility maximizing agent who lives in the model economy that now lasts for the period $[0, 2]$. New information about the true state of the economy arrives at dates $n \in \{0, 1, 2\}$. At date $n = 1$, two states of the world are possible, whereas at date $n = 2$, four states of the world are possible, $\Omega = \{uu, ud, du, dd\}$. At dates $n \in \{0, 1\}$, the agent can trade in a stock and a bond which together complete markets, as we will see. The price process of the stock is given as follows,

first date	intermediate date	terminal date
		$S_2^1(uu) = 12$
	$S_1^1(u) = 11$	
		$S_2^1(ud) = 10$
$S_0^1 = 10$		
		$S_2^1(du) = 10$
	$S_1^1(d) = 9$	
		$S_2^1(dd) = 8$
$n = 0$	$n = 1$	$n = 2$

.

Up- and down-movements happen with probability $\frac{1}{3}$ and $\frac{2}{3}$, respectively, implying $\mathbf{P}(uu) = \frac{1}{9}, \mathbf{P}(ud) = \frac{2}{9}, \mathbf{P}(du) = \frac{2}{9}$ and $\mathbf{P}(dd) = \frac{4}{9}$. The price of the bond is normalized to 1 so that the risk-less interest rate equals zero. The unique $\mathbf{P}-$equivalent martingale measure $\mathbf{Q}$ is easily computed to satisfy $\forall \omega \in \Omega : \mathbf{Q}(\omega) = \frac{1}{4}$ implying completeness of the market model $\mathcal{M}_1$.

The agent derives utility from consumption at date $n = 2$ according to the utility function,

$$v : \mathbb{R}_{++} \to \mathbb{R}, w \mapsto v(w)$$
$$\text{where } v(w) = \ln w.$$

w denotes actual consumption at date $n = 2$. The agent maximizes his / her expected utility over $n = 2$ consumption so that, given an initial endowment of the agent of $W_0 = 10$, the problem is to,

$$\max_{W_2 \in \mathbb{R}^4_{++}} \mathbf{E}_0^{\mathbf{P}}[\ln W_2] \tag{4.13}$$

$$\text{s.t. } \mathbf{E}_0^{\mathbf{Q}}[W_2] = W_0. \tag{4.14}$$

Again the budget constraint (4.14) is binding. It states that the agent can choose among those state-contingent consumption payoffs W_2 that are affordable, i.e., that cost W_0. With the respective parameter specifications one obtains from problem (4.13) and (4.14),

$$\begin{aligned} \max_{W_2(uu),W_2(ud),W_2(du),W_2(dd)} & \frac{1}{9} \cdot \ln W_2(uu) + \frac{2}{9} \cdot \ln W_2(ud) \\ & + \frac{2}{9} \cdot \ln W_2(du) + \frac{4}{9} \cdot \ln W_2(dd) \\ \text{s.t. } \frac{1}{4} \cdot W_2(uu) + \frac{1}{4} \cdot W_2(ud) + & \frac{1}{4} \cdot W_2(du) + \frac{1}{4} \cdot W_2(dd) = 10 \end{aligned}$$

The notation $W_2(\cdot)$ is consistent with that of the stock prices. Hence, the corresponding LAGRANGIAN function is,

$$\begin{aligned} & \mathcal{L}(W_2(uu), W_2(ud), W_2(du), W_2(dd), \lambda) \\ = & \frac{1}{9} \cdot \ln W_2(uu) + \frac{2}{9} \cdot \ln W_2(ud) \\ & + \frac{2}{9} \cdot \ln W_2(du) + \frac{4}{9} \cdot \ln W_2(dd) \\ & + \lambda \cdot \left(10 - \frac{1}{4} \cdot W_2(uu) - \frac{1}{4} \cdot W_2(ud) \right. \\ & \left. - \frac{1}{4} \cdot W_2(du) - \frac{1}{4} \cdot W_2(dd) \right). \end{aligned}$$

The first order conditions, which are both necessary and sufficient, are ,

$$\begin{cases} \frac{\partial \mathcal{L}}{\partial W_2(uu)} = \frac{1}{9 \cdot W_2(uu)} - \frac{1}{4} \cdot \lambda \overset{!}{=} 0 \\ \frac{\partial \mathcal{L}}{\partial W_2(ud)} = \frac{2}{9 \cdot W_2(ud)} - \frac{1}{4} \cdot \lambda \overset{!}{=} 0 \\ \frac{\partial \mathcal{L}}{\partial W_2(du)} = \frac{2}{9 \cdot W_2(du)} - \frac{1}{4} \cdot \lambda \overset{!}{=} 0 \\ \frac{\partial \mathcal{L}}{\partial W_2(dd)} = \frac{4}{9 \cdot W_2(dd)} - \frac{1}{4} \cdot \lambda \overset{!}{=} 0 \\ \frac{\partial \mathcal{L}}{\partial \lambda} \overset{!}{=} 0 \Rightarrow (4.14) \end{cases} .$$

Simple manipulations yield,

$$\begin{aligned} W_2(uu) &= \frac{1}{2} \cdot W_2(ud), \\ W_2(du) &= W_2(ud) \text{ and} \\ W_2(dd) &= 2 \cdot W_2(ud). \end{aligned}$$

Substituting for these expressions in the budget constraint eventually gives $W_2 = (W_2(uu), W_2(ud), W_2(du), W_2(dd)) = (4.\overline{4}, 8.\overline{8}, 8.\overline{8}, 17.\overline{7})$ as the optimal solution and 2.42 as the approximate optimal objective value. On this basis, one derives the optimal trading strategy as follows,

$$\begin{aligned} &\begin{cases} \phi_2^0(u) + \phi_2^1(u) \cdot 12 = 4.\overline{4} \\ \phi_2^0(u) + \phi_2^1(u) \cdot 10 = 8.\overline{8} \end{cases} \\ \Rightarrow\ &\left(\phi_2^0(u), \phi_2^1(u)\right) = (31.\overline{1}, -2.\overline{2}) \end{aligned}$$

$$\begin{aligned} &\begin{cases} \phi_2^0(d) + \phi_2^1(d) \cdot 10 = 8.\overline{8} \\ \phi_2^0(d) + \phi_2^1(d) \cdot 8 = 17.\overline{7} \end{cases} \\ \Rightarrow\ &\left(\phi_2^0(d), \phi_2^1(d)\right) = \left(53.\overline{3}, -4.\overline{4}\right) \end{aligned}$$

$\left(\phi_2^0(u), \phi_2^1(u)\right)$ denotes the optimal portfolio for the time interval $[1, 2]$ in states uu and ud. Accordingly, $\left(\phi_2^0(d), \phi_2^1(d)\right)$ denotes the optimal portfolio for the time interval $[1, 2]$ in states du and dd. Moreover, wealth of the agent at $n = 1$ is $(W_1(u), W_1(d)) = (6.\overline{6}, 13.\overline{3})$, where the notation is again in line with that of the stock prices. And from this emerges,

$$\begin{aligned} &\begin{cases} \phi_1^0 + \phi_1^1 \cdot 11 = 6.\overline{6} \\ \phi_1^0 + \phi_1^1 \cdot 9 = 13.\overline{3} \end{cases} \\ \Rightarrow\ &\left(\phi_1^0, \phi_1^1\right) = (43.\overline{3}, -3.\overline{3}). \end{aligned}$$

$\left(\phi_1^0, \phi_1^1\right)$ denotes the optimal portfolio during the time interval $[0, 1[$.

4.5.2 Optimal consumption in incomplete markets

Consider now the agent of the previous example living in an economy where he / she faces in the market model $\mathcal{M}_1$ a stock price process of,

first date	intermediate date	terminal date
		$S_2^1(uu) = 12$
	$S_1^1(u) = 11$	$S_2^1(um) = 11$
		$S_2^1(ud) = 10$
$S_0^1 = 10$		
		$S_2^1(du) = 10$
	$S_1^1(d) = 9$	$S_2^1(dm) = 9$
		$S_2^1(dd) = 8$
$n = 0$	$n = 1$	$n = 2$

Obviously, the state space has enlarged to $\Omega = \{uu, um, ud, du, dm, dd\}$. Assume that the probabilities for the three states uu, um and ud to unfold are respectively $\frac{1}{9}$. Assume further that the probabilities for the three other states are $\frac{1}{9}$ for du, $\frac{3}{9}$ for dm and $\frac{2}{9}$ for dd. From the definition of a martingale,

$$\begin{aligned} S_0^1 &\overset{!}{=} \mathbf{E}_0^{\mathbf{Q}}[S_1^1] \\ \Leftrightarrow 10 &= q_u \cdot 11 + (1 - q_u) \cdot 9, \end{aligned}$$

giving unique transition probabilities at date 0 of $q_u = 0.5 \equiv \mathbf{Q}(u)$ and $1 - q_u = 0.5 \equiv \mathbf{Q}(d)$. In contrast, the martingale definition fails to deliver unique transition probabilities at date $n = 1$. This can be seen from,

$$\begin{aligned} S_1^1(u) &\overset{!}{=} \mathbf{E}_{1u}^{\mathbf{Q}}[S_2^1] \\ \Leftrightarrow 11 &= q_{uu} \cdot 12 + q_{um} \cdot 11 + q_{ud} \cdot 12, \end{aligned}$$

and,

$$\begin{aligned} S_1^1(d) &\overset{!}{=} \mathbf{E}_{1d}^{\mathbf{Q}}[S_2^1] \\ \Leftrightarrow 9 &= q_{du} \cdot 10 + q_{dm} \cdot 9 + q_{dd} \cdot 8. \end{aligned}$$

$\mathbf{E}_{1u}^{\mathbf{Q}}[\cdot]$ and $\mathbf{E}_{1d}^{\mathbf{Q}}[\cdot]$ denote conditional expectation taken at the u node and the d node at $n = 1$, respectively. Using the same calculation scheme as in sub-section 4.4.2, one can identify the following solutions,

$$\begin{cases} q_{um} = 1 - 2 \cdot q_{uu} \\ q_{ud} = q_{uu} \\ q_{uu} \in \left]0, \frac{1}{2}\right[, \end{cases}$$

and,

$$\begin{cases} q_{dm} = 1 - 2 \cdot q_{du} \\ q_{dd} = q_{du} \\ q_{du} \in \left]0, \frac{1}{2}\right[\end{cases}.$$

Together this gives us the set of all $\mathbf{P}-$equivalent martingale measures as,

$$\begin{aligned} \mathbb{Q} &= \left\{ \mathbf{Q} \in \mathbb{R}^6_{++} : \mathbf{Q} = \frac{1}{2} \cdot (\rho, 1 - 2 \cdot \rho, \rho, \rho', 1 - 2 \cdot \rho', \rho') \right. \\ &\qquad \left. \text{and } \rho, \rho' \in \left]0, \frac{1}{2}\right[\right\}. \end{aligned}$$

The over-countably many probability measures contained in $\mathbb{Q}$ imply by proposition 35 that the market model $\mathcal{M}_1$ is now incomplete. The set of attainable contingent claims is,

$$\mathbb{A} = \left\{ A_2 \in \mathbb{R}^6_+ : \begin{array}{l} A_2 = \nu \cdot \begin{pmatrix} 12 \\ 11 \\ 10 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \nu' \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \widetilde{\nu} \cdot \begin{pmatrix} 0 \\ 0 \\ 0 \\ 10 \\ 9 \\ 8 \end{pmatrix} + \widetilde{\nu}' \cdot \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \\ \nu, \nu', \widetilde{\nu}, \widetilde{\nu}' \in \mathbb{R} \end{array} \right\},$$

Obviously, the set of attainable contingent claims $\mathbb{A}$ is a proper subset of $\mathbb{R}^6_+$ as expected.

It can be checked that,

$$\mathbb{Q}^B = \left\{ \mathbf{Q}^B_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \mathbf{Q}^B_2 = \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{pmatrix}, \mathbf{Q}^B_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \mathbf{Q}^B_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{pmatrix} \right\},$$

forms a martingale basis. To verify this claim, let ρ and ρ' be fixed. The linear system,

$$\theta_1 \cdot \mathbf{Q}^B_1 + \theta_2 \cdot \mathbf{Q}^B_2 + \theta_3 \cdot \mathbf{Q}^B_3 + \theta_4 \cdot \mathbf{Q}^B_4 = \mathbf{Q}(\rho, \rho'),$$

yields as unique solutions $\theta_1 = \frac{1}{2} - \rho, \theta_2 = \rho$, $\theta_3 = \frac{1}{2} - \rho'$ and $\theta_4 = \rho'$, proving that $\mathbb{Q}^B$ is indeed a martingale basis.

In this context, the problem of the agent is to,

$$\max_{W_2 \in \mathbb{A}} \mathbf{E}_0^{\mathbf{P}}[\ln W_2] \tag{4.15}$$

$$\text{s.t. } \mathbf{E}_0^{\mathbf{Q}_1^B}[W_2] = W_0 \tag{4.16}$$

$$\mathbf{E}_0^{\mathbf{Q}_2^B}[W_2] = W_0 \tag{4.17}$$

$$\mathbf{E}_0^{\mathbf{Q}_3^B}[W_2] = W_0 \tag{4.18}$$

$$\mathbf{E}_0^{\mathbf{Q}_4^B}[W_2] = W_0, \tag{4.19}$$

or more detailed it is to,

$$\begin{aligned} \max_{W_2 \in \mathbb{A}} \mathbf{E}_0^{\mathbf{P}}[\ln W_2] \\ \text{s.t.} W_2(um) = 10 \\ \frac{1}{2} \cdot W_2(uu) + \frac{1}{2} \cdot W_2(ud) = 10 \\ W_2(dm) = 10 \\ \frac{1}{2} \cdot W_2(du) + \frac{1}{2} \cdot W_2(dd) = 10. \end{aligned}$$

The first order conditions of the corresponding LAGRANGIAN function are,

$$\begin{cases} \frac{\partial \mathcal{L}}{\partial W_2(uu)} = \frac{1}{9 \cdot W_2(uu)} - \frac{1}{2} \cdot \lambda_2 \stackrel{!}{=} 0 \\ \frac{\partial \mathcal{L}}{\partial W_2(um)} = \frac{1}{9 \cdot W_2(um)} - \lambda_1 \stackrel{!}{=} 0 \\ \frac{\partial \mathcal{L}}{\partial W_2(ud)} = \frac{1}{9 \cdot W_2(ud)} - \frac{1}{2} \cdot \lambda_2 \stackrel{!}{=} 0 \\ \frac{\partial \mathcal{L}}{\partial W_2(du)} = \frac{1}{9 \cdot W_2(du)} - \frac{1}{2} \cdot \lambda_3 \stackrel{!}{=} 0 \\ \frac{\partial \mathcal{L}}{\partial W_2(dm)} = \frac{3}{9 \cdot W_2(dm)} - \lambda_4 \stackrel{!}{=} 0 \\ \frac{\partial \mathcal{L}}{\partial W_2(dd)} = \frac{2}{9 \cdot W_2(dd)} - \frac{1}{2} \cdot \lambda_3 \stackrel{!}{=} 0 \\ \forall i : \frac{\partial \mathcal{L}}{\partial \lambda_i} \stackrel{!}{=} 0 \Rightarrow \text{(4.16)-(4.19)} \end{cases}$$

We obtain,

$$\begin{aligned} W_2(uu) &= W_2(ud), \\ W_2(um) &= 10, \\ W_2(dd) &= 2 \cdot W_2(du) \text{ and} \\ W_2(dm) &= 10, \end{aligned}$$

from which we deduce with the help of the budget constraints (4.17) and (4.19) that in optimum $W_2(uu) = W_2(um) = W_2(ud) = W_2(dm) = 10$,

$W_2(du) = \frac{20}{3}$ and $W_2(dd) = \frac{40}{3}$, i.e., $W_2 = \left(10, 10, 10, \frac{20}{3}, 10, \frac{40}{3}\right)$. The corresponding optimal objective value is 2.32. The different portfolios that correspond to the trading strategy generating the optimal state-contingent payoff are determined as,

$$\begin{aligned}
\left(\phi_1^0, \phi_1^1\right) &= (10, 0), \\
\left(\phi_2^0(u), \phi_2^1(u)\right) &= (10, 0), \\
\left(\phi_2^0(d), \phi_2^1(d)\right) &= \left(40, -\frac{10}{3}\right),
\end{aligned}$$

where we used the notation of the previous example. The wealth of the agent at date 1 is $(W_1(u), W_1(d)) = (10, 10)$.

4.6 Summary

In economic terms, the following four findings of section 4.3 seem particularly important:

1. We saw that there is a direct relationship between price systems and equivalent martingale measures in the market model $\mathcal{M}$.

2. We also saw that the absence of arbitrage implies that the set of equivalent martingale measures and the set of consistent linear pricing systems are non-empty and vice versa.

3. Maximization problems of expected utility maximizing agents have solutions if there is no arbitrage, there is an equivalent martingale measure or there is a consistent linear price system.

4. The market model $\mathcal{M}$ is complete, thereby implying that every contingent claim is attainable via an admissible trading strategy, if the set of equivalent martingale measures is a singleton.

The martingale approach provides useful means with which we can conveniently tackle contingent claim pricing and decision making problems. Below we summarize several aspects of the martingale approach which are important in methodical terms. Regarding contingent claim pricing and optimal consumption problems, section 4.3 and section 4.4 revealed the following:

1. Contingent claim pricing:

    (a) In *complete* markets, the price of an arbitrary contingent claim is merely the expectation of its state-contingent payoff under the *unique* equivalent martingale measure.[15]

    (b) In *incomplete* markets, the price of an attainable contingent claim equals the expectation of its state-contingent payoff under *any* equivalent martingale measure. However, in applications it is often useful to work with a martingale basis instead of the whole set $\mathbb{Q}$, thereby considerably reducing the number of 'relevant' martingale measures.

2. Optimal consumption problems:

    (a) Using martingale methods, the budget constraint of an expected utility maximizing agent in *complete* markets says that the expectation of the state-contingent consumption payoff under the *unique* equivalent martingale measure must equal the initial wealth.

    (b) In *incomplete* markets, there are generally more than one budget constraint. With markets being incomplete, the expectation of the state-contingent consumption payoff under *each* equivalent martingale measures must equal the initial wealth of the agent. Making use of a martingale basis, the number of budget constraints coincides with the (finite) number of elements in the martingale basis.

In dynamic settings, optimal consumption problems may be transformed into static ones. The application of martingale methods enables the separation of the task of identifying the optimal solution, i.e., the optimal state-contingent consumption payoff from the task of determining the corresponding trading strategy generating the optimal state-contingent payoff. In contrast, when applying dynamic programming techniques, both tasks are closely intertwined. The single steps of the three date examples of section 4.5 can be put together to form a 'cookbook recipe':

[15] For the moment, we abstract from interest rate related issues. If interest rates were explicit, a need for discounting would arise of course.

1. Solve the optimization problem in martingale form and obtain as output an optimal state-contingent consumption payoff for the terminal date.

2. Equipped with this optimal payoff, work backwards in time to completely derive the optimal wealth process and the trading strategy that generates it.[16]

[16]Since all problems analyzed in this thesis are terminal consumption problems, we abstract here from optimal consumption problems where agents may consume before the terminal date of the economy. Of course, the recipe only applies to terminal consumption problems.

Part III

Applications

In this last part of the thesis, we apply the martingale methods presented in part II of the thesis to three different economic settings. The first application is to the binomial model of Cox, Ross, and Rubinstein (1979). Within their framework, we examine in chapter 5 dynamic hedging strategies. We give a proof for the claim that dynamic hedging of contingent claims with convex payoffs leads to positive feedback.

The other two applications are to general equilibrium models. In chapter 6 and 7, a general equilibrium analysis of dynamic hedging is carried out. The market model considered in chapter 6 is complete in equilibrium whereas inherent market incompleteness characterizes the market model considered in chapter 7. Apart from this exception, the two models share all other features. Both are, for instance, representable by event trees. One of the early treatments of equilibrium models with uncertainty being represented by event trees is found in Debreu (1959, chapter 7). Arrow (1964) was the first to emphasize the spanning role of securities in such a context. Another assumption characterizing both chapters is that there is a continuum of agents populating the model economy, a formulation dating back to Aumann (1964) and Aumann (1966). In the spirit of similar noise trader studies, the population itself comprises two types of agents: rational agents (non-hedgers) and noise traders (hedgers).

Questions that naturally arise in a general equilibrium context are those of the existence and the determinacy of general equilibria. Answers to these questions depend in a crucial manner on whether markets are complete or not. Regarding the existence of general equilibria in complete markets results may be found in the pioneering texts of Arrow and Debreu (1954), Debreu (1959), and Arrow and Hahn (1971). Duffie and Sonnenschein (1989), for example, provide a survey of existence results and give many further references to early and more recent work. The study of the existence of general equilibrium in incomplete markets is a newer discipline. Surveys are given, for instance, in Duffie (1992) and Magill and Quinzii (1996).

Our approach to investigating the impact of dynamic hedging on market equilibrium heavily relies on comparative statics analysis. Considering this, the uniqueness of general equilibrium is a condition we cannot dispense with. In this respect, Duffie and Sonnenschein (1989, 575) remark:

> "In the absence of uniqueness, the comparative statics of how prices and allocations will change with a change in the parameter values is not a well-defined exercise."

Arrow and Hahn (1971, chapter 9) and Kehoe (1985) survey results regarding the determinacy of general equilibrium in complete markets while

CASS (1992) surveys those obtained in incomplete markets.[17] Unfortunately, necessary assumptions that ensure the uniqueness of a general equilibrium are very strong in general.

The way by which we guarantee uniqueness is to postulate the existence of a representative agent or, more precisely, to postulate that all rational agents are identical and satisfy certain conditions relating to their utility function. Although it may seem quite restrictive, it is nonetheless a very common approach to asset pricing, as DUFFIE (1992, 230) notes:

> "The traditional approach to asset pricing theory has been to assume either a single agent or complete markets."

This statement should be judged in the light of results by CONSTANTINIDES (1982) who proves the existence of a representative agent in complete markets. Since the representative agent paradigm applies to complete as well as to incomplete markets equally well, we directly assume the existence of such an agent. This allows us to pursue a rather general strategy in exploring the general equilibrium models because, in principle, we do not have to differentiate between complete markets settings and incomplete ones. In fact, the analysis of the representative agent's problem is very similar in both complete and incomplete markets. MILNE (1995) provides a rather comprehensive survey of representative agent theory.

Another strand of literature related to our approach is concerned with the viability of equilibrium price processes. BICK (1987), BICK (1990) and HE and LELAND (1993) derive necessary and sufficient conditions for the viability of equilibrium price processes in complete markets. In a sense, we extend upon their work by showing that some results carry over to settings where noise traders are present and also to settings with inherent market incompleteness.

[17] An interesting treatment of this issue may also be found in CASS (1991).

Chapter 5

Dynamic hedging and positive feedback

5.1 Introduction

The plan for this chapter is roughly as follows. We first explore in section 5.2 and sub-section 5.3.1 the binomial option pricing model originally proposed in COX, ROSS, and RUBINSTEIN (1979). Second, we compare in sub-section 5.3.2 the binomial pricing model with the approach of BLACK and SCHOLES (1973) to pricing options. We outline that there exists a close relationship due to strong convergence results from discrete to continuous time. Third, we will turn in section 5.4 to dynamic hedging strategies and prove that they produce positive feedback if the corresponding contingent claim has a convex state-contingent payoff. With regard to the positive feedback result, an example based on the BLACK / SCHOLES option pricing formula will provide a graphic illustration. A brief summary in section 5.5 concludes the chapter.

5.2 The market model

This section embeds the binomial option pricing model into the general framework introduced in chapter 4. Our exposition follows that in LAMBERTON and LAPEYRE (1996, section 1.4) rather closely. However, the original article is quite accessible as well and the model itself has become a commonplace in any good finance book.

In the COX / ROSS / RUBINSTEIN model, there are only two securities, one of which is risky, called *stock*, while the other one is risk-less and called *bond*. Therefore, we have the special case where the number of risky securities is $K = 1$ in the market model $\mathcal{M} = \{(\Omega, \wp(\Omega), \mathbb{F}, \mathbf{P}), N, \mathbb{S}, \mathbb{I}\}$ of section 4.2. Apart from this exception, we employ all other assumptions made in section 4.2, and in particular the standard assumptions as found in section 1.1. In the following, we will further specify several elements of the market model.

The model economy lasts for the time period $[0, N]$, where $N \in \mathbb{N}$ and $1 < N < \infty$. New information arrives at $N + 1$ dates $n \in \{0, 1, ..., N\}$. At these dates only, economic activity is observed. The bond, throughout the chapter indicated by the superscript '0', has a date n price S_n^0 that is given by,

$$S_n^0 \equiv (1 + r)^n.$$

r denotes the constant, risk-less interest rate for all time intervals $[n, n + 1[$, $n \in \{0, ..., N - 1\}$. Thus, the discount process $(\beta_n)_{n\in\{0,...,N\}}$ satisfies $\forall n$: $\beta_n \equiv (1 + r)^{-n}$.

We label the stock with the superscript '1' throughout the chapter. Given the stock price S_n^1 at date $n \in \{0, ..., N - 1\}$ the stock price at date $n + 1$ is,

$$S_{n+1}^1 \equiv S_n^1 \cdot m,$$

where nature chooses m randomly out of $\{1 + u, 1 + d\}$. u and d are constant and satisfy $-1 < d < u$. S_0^1 is strictly positive, fixed and publicly known. Security prices S_n^0 and $S_n^1, n \in \{0, ..., N - 1\}$, are constant over the time interval $[n, n + 1[$.

Uncertainty in the model economy is generated by the stock price process $(S_n^1)_{n\in\{0,...,N\}}$. In particular, the stock price process gives rise to $\Omega = \{1 + u, 1 + d\}^N$. In other words, to each state ω of the economy at date N corresponds a stock price path with $j \geq 0$ upward-movements $1 + u$, and $N - j$ downward-movements $1 + d$.[1] These paths may formally be represented by date-ordered $N-$tuples $(m_1, ..., m_N)$ where $\forall n$: $m_n \in \{1 + u, 1 + d\}$. A graphic illustration will be given shortly. The filtration $\mathbb{F} = (\mathcal{F}_n)_{n\in\{0,...,N\}}$ is the filtration generated by the stock price process $(S_n^1)_{n\in\{0,...,N\}}$. As a consequence, it satisfies $\mathcal{F}_0 \equiv \{\varnothing, \Omega\}$, $\mathcal{F}_N \equiv \wp(\Omega)$ and $\mathcal{F}_n \equiv \sigma(S_i : i \in \{0, ..., n\})$. Moreover, we assume that the probability measure $\mathbf{P}$ satisfies $\forall \omega \in \Omega$: $\mathbf{P}(\omega) > 0$. Taking everything together, this defines the filtered probability space $(\Omega, \wp(\Omega), \mathbb{F}, \mathbf{P})$ with which we will work.

We now want to provide a graphic illustration for the resolution of uncertainty in the binomial model. An appropriate tool for this task is the event

[1] Since it is quite common in the literature, we use the words 'upward-movement' (for $1+u$) and 'downward-movement' (for $1+d$) *relative* to the risk-less return $(1+r)$. If fact, it can also be that $1 + d > 1$ meaning that even a 'downward-movement' leads to a higher stock price.

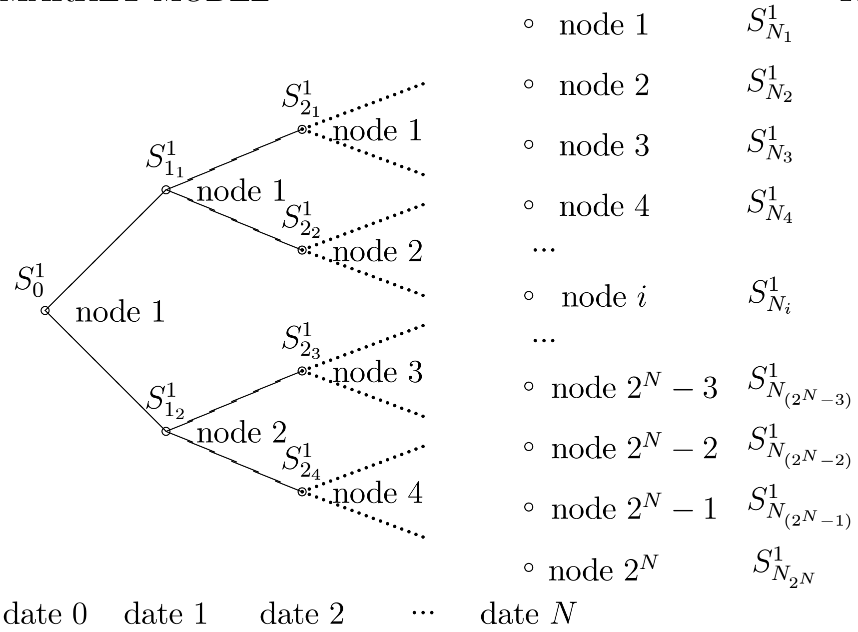

Figure 5.1: The event tree generated by the stock price process.

tree.[2] An event tree describing the binomial model is depicted in figure 5.1. As can be seen in this figure, the stock price process generates an event tree with 2^n different nodes at date $n \in \{0, ..., N\}$.[3] Consequently, the number $|\Omega|$ of possible states of the economy at date N equals the number of terminal nodes 2^N since each state of the economy corresponds to exactly one terminal node of the tree. If one assigns a top-down numbering to all nodes at a given date n, one can denote by n_i, $i \in \{1, 2, ..., 2^n\}$, the $i-$th node from the top at date n. The stock prices corresponding to these nodes are denoted by $S_{n_i}^1, i \in \{1, 2, ..., 2^n\}$.

Summary

In summary, we have the market model,

$$\mathcal{M}^{CRR} = \{(\Omega, \wp(\Omega), \mathbb{F}, \mathbf{P}), N, \mathbb{S}^1, \mathbb{I}\},$$

where,

---

[2]Formally, the type of tree we consider is characterized by two properties: (i) the tree has exactly one node that has no predecessor and (ii) every node in the tree has exactly one predecessor. In economic analyses, this type of tree is sometimes also called *information tree* or *decision tree*. Section 12 of DUFFIE (1988) contains a general treatment of economies that allow for an event tree representation.

[3]At date $n = 0$ there is only one ($2^0 = 1$) node, at date $n = 1$ there are two ($2^1 = 2$), at date $n = 2$ there are already four ($2^2 = 4$) and so forth.

- $\Omega = \{1 + u, 1 + d\}^N$,
- $\mathbb{F}$ is the filtration generated by the stock price process $(S_n^1)_{n\in\{0,\ldots,N\}}$,
- $\mathbf{P}$ is strictly positive for all $\omega \in \Omega$,
- $N \in \mathbb{N}$ satisfies $1 < N < \infty$,
- $\mathbb{S}^1 \equiv \{(S_n^k)_{n\in\{0,\ldots,N\}} : k \in \{0, 1\}\}$ and
- $\mathbb{I}$ is as defined as in chapter 4.

5.3 Contingent claim pricing

5.3.1 Pricing in the Cox, Ross, and Rubinstein (1979) model

With the complete setup $\mathcal{M}^{CRR}$ of the Cox / Ross / Rubinstein model we can go on and derive prices for contingent claims. The sub-section culminates in the famous pricing result for European call options, namely the binomial option pricing formula. Before we attack this pricing formula, it is worthwhile to consider some economic aspects relating to the present market model first.

Lemma 36 provides a necessary condition for $\mathcal{M}^{CRR}$ to be arbitrage-free. It will turn out, however, that it is also sufficient.

Lemma 36 *If the market model,*

$$\mathcal{M}^{CRR} = \{(\Omega, \wp(\Omega), \mathbb{F}, \mathbf{P}), N, \mathbb{S}^1, \mathbb{I}\},$$

is arbitrage-free, then $r \in]d, u[$.

Proof. In arbitrage-free markets there exists a $\mathbf{P}$-equivalent probability measure $\mathbf{Q}$ that makes the discounted stock price process $(\beta_n S_n^1)_{n\in\{0,\ldots,N\}}$ a martingale. This result has been formulated as part of Theorem 29. Under such a $\mathbf{Q}$, it must therefore hold for all $n \leq N - 1$ that,

$$\begin{aligned}
\mathbf{E}_n^{\mathbf{Q}}[\beta_{n+1} \cdot S_{n+1}^1] &= \beta_n \cdot S_n^1 \\
\Leftrightarrow \mathbf{E}_n^{\mathbf{Q}}\left[\frac{S_{n+1}^1}{S_n^1}\right] &= \frac{\beta_n}{\beta_{n+1}} \\
\Leftrightarrow \mathbf{E}_n^{\mathbf{Q}}\left[\frac{S_{n+1}^1}{S_n^1}\right] &= 1 + r.
\end{aligned}$$

And since $\frac{S^1_{n+1}}{S^1_n} \in \{1+u, 1+d\}$, it follows that $1+r \in]1+d, 1+u[$ and so $r \in]d, u[$ as asserted. Also compare LAMBERTON and LAPEYRE (1996, 12). ■

In view of lemma 36, let us assume for this chapter that $r \in]d, u[$. One then obtains,

Lemma 37 *In the market model* $\mathcal{M}^{CRR} = \{(\Omega, \wp(\Omega), \mathbb{F}, \mathbf{P}), N, \mathbb{S}^1, \mathbb{I}\}$, *there exists a unique* $\mathbf{P}-$*equivalent martingale measure, henceforth denoted* $\mathbf{Q}$, *that makes the discounted stock price process* $(\beta_n S^1_n)_{n\in\{0,\ldots,N\}}$ *a martingale. The unique transition probabilities at all nodes and dates associated with this martingale measure are* $q = \frac{r-d}{u-d}$ *for an upward-movement* $1+u$, *and* $1-q$ *for a downward-movement* $1+d$.

Proof. Note that the discounted stock price process $(\beta_n S^1_n)_{n\in\{0,\ldots,N\}}$ is a martingale under a probability measure $\mathbf{Q}$ if for all $n \leq N-1$,

$$\begin{aligned}
\mathbf{E}^{\mathbf{Q}}_n[\beta_{n+1} \cdot S^1_{n+1}] &= \beta_n \cdot S^1_n \\
\Leftrightarrow \mathbf{E}^{\mathbf{Q}}_n\left[\frac{S^1_{n+1}}{S^1_n}\right] &= \frac{\beta_n}{\beta_{n+1}} \\
\Rightarrow q \cdot (1+u) + (1-q) \cdot (1+d) &= 1+r \\
\Leftrightarrow q &= \frac{r-d}{u-d}.
\end{aligned}$$

This proves the lemma since the unique transition probabilities q and $(1-q)$ define the unique $\mathbf{P}-$equivalent martingale measure $\mathbf{Q}$. ■

Proposition 38 *The market model* $\mathcal{M}^{CRR} = \{(\Omega, \wp(\Omega), \mathbb{F}, \mathbf{P}), N, \mathbb{S}^1, \mathbb{I}\}$ *is arbitrage-free and complete. Formally,* $\mathbb{A} = \mathbb{R}^{|\Omega|}_+$.

Proof. Lemma 37 is the key to the proof. Arbitrage-freeness follows from lemma 37 and Theorem 29. Arbitrage-freeness, lemma 37 and proposition 35 together imply completeness. ■

After having ensured that the market model is complete, we can be sure that every contingent claim is attainable. Furthermore, arbitrage-freeness implies that it is possible to associate a unique price with every contingent claim.

The contingent claims that are of particular interest for the moment are European call options. Recall that European call options on a stock give its holder the right but not the obligation to buy the underlying stock at maturity - in our model at date N - for a predetermined price X, the so-called exercise price. The payoff C_N of such an option contract with exercise price X and maturity N is formally given by[4],

$$C_N : \Omega \to \mathbb{R}_+, \omega \mapsto \max\{S_N^1(\omega) - X, 0\}. \tag{5.1}$$

For such an option, we have,

Proposition 39 (Cox, Ross, **and** Rubinstein **(1979)**) *In the market model* $\mathcal{M}^{CRR} = \{(\Omega, \wp(\Omega), \mathbb{F}, \mathbf{P}), N, \mathbb{S}^1, \mathbb{I}\}$*, the date n price of a European call option described by (5.1) is,*

$$C_n(S_n^1) = (1+r)^{-(N-n)} \cdot \mathbf{E}_n^{\mathbf{Q}} \left[\max\left\{ S_n^1 \cdot \prod_{i=n+1}^{N} \frac{S_i^1}{S_{i-1}^1} - X, 0 \right\} \right] \tag{5.2}$$

$$= (1+r)^{-(N-n)} \cdot \sum_{j=0}^{N-n} \left[\frac{(N-n)!}{(N-n-j)!j!} \cdot q^j \cdot (1-q)^{N-n-j} \right.$$
$$\left. \cdot \max\left\{ S_n^1 \cdot (1+d)^j \cdot (1+u)^{N-n-j} - X, 0 \right\} \right], \tag{5.3}$$

where everything is as defined as before.

Proof. (5.2) is an application of corollary 30 and proposition 31. For a proof of (5.3) refer to Cox, Ross, and Rubinstein (1979). After obvious notational adjustments, (5.3) corresponds to formula (6) in that article. ■

Remark 7 *We can calculate the date n price P_n of a European put option with the same defining properties as the call with the help of the put-call parity,*

$$C_n(S_n^1) - P_n(S_n^1) = S_n^1 - X \cdot (1+r)^{-(N-n)}.$$

5.3.2 Comparison with Black and Scholes (1973)

The model of Cox, Ross, and Rubinstein (1979) and their derivation of the binomial option pricing formula (5.3) have marked an important step towards a deeper understanding of the principles underlying the Black / Scholes / Merton approach to option pricing.[5] The concept of risk-less arbitrage is something that is almost visible and intuitively accessible in the binomial option pricing model. In contrast, in the Black / Scholes model risk-less arbitrage is something that happens behind the scenes since they

[4] See also the example in sub-section 4.4.1.

[5] Also refer to the informal discussion in section 1.1.

work in continuous time.[6] The tractability and transparency of the discrete time model is one major reason why the binomial model has become very popular in the financial services industry. HE (1990, 523-524) remarks:

> "The binomial model provides an easy way of explaining (without using advanced mathematics) how uncertainties are resolved in the continuous-time model and how continuous trading in the stock and the bond can span infinitely many states of nature. More importantly, it provides an elegant numerical alternative to the partial differential equations (PDE) obtained in continuous-time models. The binomial option-pricing technique has now become an extremely powerful tool for valuing derivative securities that might be difficult to price under other alternative methods."

Nevertheless, working in discrete time also produces disadvantages. Generally, there are no closed form solutions available that could be manipulated analytically. Therefore, we want to state the original result of BLACK and SCHOLES (1973) although it does not fit very well in the mathematical setup of the chapter and the thesis as a whole. However, because of its closed form-property, it enables graphic illustrations of some aspects we will tackle in section 5.4. This excursion to a continuous time setting will hopefully help in the end to better grasp the ideas behind the rather formal considerations that we will conduct in section 5.4.

Since working in continuous time, BLACK and SCHOLES (1973) consider all points of a given time interval $[0, T]$ to be relevant instead of a finite selection of dates out of this interval. In their model, there is a continuous inflow of new information. To avoid ambiguity, we label dates by t where $t \in [0, T]$ when referring to the BLACK / SCHOLES world.[7] With this notation, a European call option is defined by[8],

$$C_T : \mathbb{R}_{++} \to \mathbb{R}_+, S_T^1 \mapsto \max\{S_T^1 - X, 0\}. \tag{5.4}$$

Now,

[6] The BLACK / SCHOLES pricing formula is often described by the picture of a 'black box' because one can only observe *how* a change in the input variables changes the output, i.e., the option price, but not *why* it does so.

[7] In the BLACK / SCHOLES world, the stock price follows a geometric BROWNIAN motion. For the other assumptions characterizing the BLACK / SCHOLES world, refer to the original article BLACK and SCHOLES (1973).

[8] Note that the call option is here defined directly on the stock price. In the BLACK / SCHOLES model, the positive real line represents the set of possible states of the economy.

Proposition 40 (Black **and** Scholes **(1973))** *The date t price of the option (5.4) is,*

$$C_t = S_t^1 \cdot \Phi(d_1) - e^{-r^{BS} \cdot t^*} \cdot X \cdot \Phi(d_2) \tag{5.5}$$

$$\text{with } d_1 \equiv \frac{\ln \frac{S_t^1}{X} + \left(r^{BS} + \frac{(\sigma^{BS})^2}{2} \right) \cdot t^*}{\sigma^{BS} \cdot \sqrt{t^*}}, \tag{5.6}$$

$$d_2 \equiv d_1 - \sigma^{BS} \cdot \sqrt{t^*}, \tag{5.7}$$

$$t^* \equiv T - t \text{ and}$$

$$\Phi(d) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d} e^{-\frac{x^2}{2}} dx.$$

S_t^1 denotes the stock price at date t, t^* the remaining life-time of the option, X the exercise price, r^{BS} and σ^{BS} the (constant) risk-less interest rate and the (constant) volatility parameter in the Black / Scholes *model, respectively.*

Proof. Wilmott, Howison, and Dewynne (1995), for instance, provide in chapter 5 a proof which draws on partial differential equations. This way of proof has been the first rigorous one to attack the option pricing problem. Briys, Bellalah, Mai, and De Varenne (1998, 53-56), for example, prove the result the martingale way. Of course, the seminal article of Black and Scholes (1973) may also be consulted. ■

Remark 8 *As mentioned in the context of the binomial option pricing formula, one can derive European put option prices P_t easily from (5.5) and the put-call parity. In continuous time, this relationship takes on the form,*

$$C_t(S_t^1) - P_t(S_t^1) = S_t^1 - X \cdot e^{-r^{BS} \cdot t^*}.$$

A numerical example shall demonstrate the graphic capabilities of formula (5.5).

Example 41 *Consider a European call option on a stock which has a volatility of $\sigma^{BS} = 0.30$ while the risk-less interest rate is $r^{BS} = 0.05$. The defining properties of the call option are $X = 100$ and $t^* = 0.75$. Figure 5.2 plots the option price against S_t^1 and t^*, figure 5.3 does it against S_t^1 and σ^{BS}. As the figures visualize, increases in S_t^1, t^* or σ^{BS} all have a positive impact on the call option's value.* ■

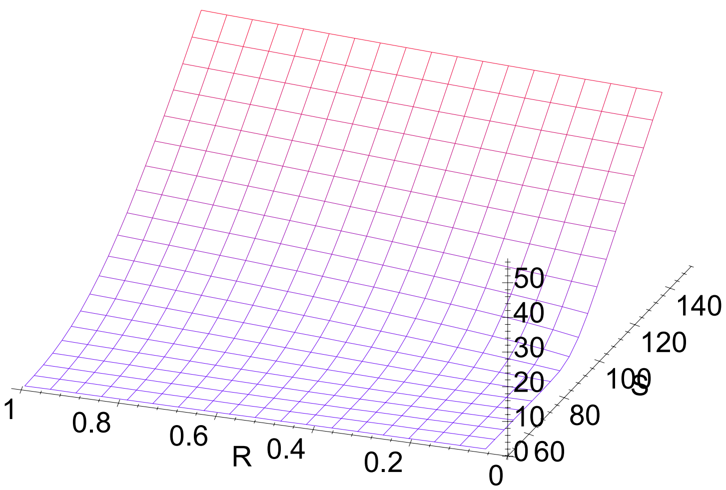

Figure 5.2: The BLACK / SCHOLES value of the call option for varying S_t^1 (denoted by S) and t^* (denoted by R).

As already pointed out in the original contribution of COX, ROSS, and RUBINSTEIN (1979), option prices obtained from the binomial pricing formula converge, under appropriate assumptions, to those obtained from the BLACK / SCHOLES formula if the number of dates N reaches infinity. Formally,

$$\lim_{N\to\infty} C_n^{CRR} = C_n^{BS},$$

where C_n^{CRR} denotes a European call option's price at date n according to COX, ROSS, and RUBINSTEIN (1979) and C_n^{BS} denotes its price according to BLACK and SCHOLES (1973) at this date. About 10 years later, HE (1990) generalizes this idea considerably in his article:

> "... we present a convergence from discrete-time multivariate multinomial models to a general continuous-time multidimensional diffusion model for contingent claim prices. ... We show that the contingent claim prices and the replicating portfolio strategies derived from the discrete-time models converge to the corresponding contingent claims prices and replicating portfolio strategies of the limiting continuous-time model." HE (1990, 524).

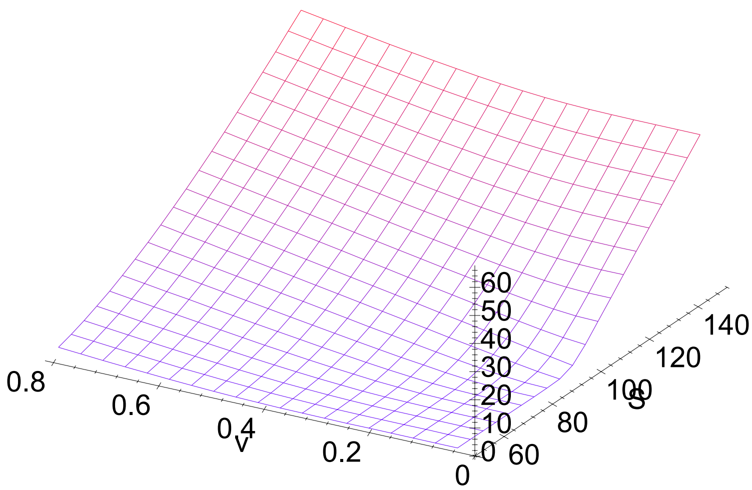

Figure 5.3: The Black / Scholes value of the call option for varying S_t^1 (denoted by S) and σ (denoted by v).

This concludes our excursion to the Black / Scholes world of option pricing. The next section returns to the binomial model $\mathcal{M}^{CRR}$ to investigate dynamic hedging strategies.

5.4 The main result

5.4.1 Derivation of the main result

Section 5.2 was primarily concerned with the derivation of reasonable prices for attainable contingent claims, i.e., prices enforced by the absence of arbitrage. According to definition 24, a contingent claim is attainable if there exists an admissible trading strategy that generates its payoff at maturity. A major part of the elegance of the martingale approach is due to the circumstance that trading strategies themselves can almost be neglected; they 'only have to exist'. However, besides the importance of the ideas regarding the pricing of contingent claims developed so far, a second field of application has become even more important in the real marketplace in recent times: *dynamic hedging of contingent claims*.[9] The remainder of this sub-section focuses on the development of the strong result that dynamic hedging of a

[9] Recall the discussion in section 1.3.

contingent claim, i.e., the implementation of a trading strategy that generates its payoff, leads to positive feedback trading if the payoff is convex. The Cox / Ross / Rubinstein model, $\mathcal{M}^{CRR} = \{(\Omega, \wp(\Omega), \mathbb{F}, \mathbf{P}), N, \mathbb{S}^1, \mathbb{I}\}$, proves to be a fruitful place for this intention.

All assumptions and notational agreements of section 5.2 remain in force. Suppose that the predictable vector process $(\phi_n)_{n\in\{0,...,N\}}$, which consists of the two component processes $(\phi_n^0)_{n\in\{0,...,N\}}$ and $(\phi_n^1)_{n\in\{0,...,N\}}$, is the admissible trading strategy that a hedger implements in order to dynamically hedge an attainable contingent claim $A_N \in \mathbb{A}$. Hence,

$$\begin{aligned} A_N &\equiv V_N(\phi) \text{ and} \\ A_0 &\equiv V_0(\phi) = \mathbf{E}_0^{\mathbf{Q}}\left[\beta_N \cdot V_N(\phi)\right]. \end{aligned}$$

We will simply refer to such a strategy as the 'dynamic hedging strategy for A_N'.

With respect to the derivation of the main result, the plan is as follows. First, we derive the number of shares held by the hedger at a given node of the tree. Second, a certain property of convex functions is stated and proved. Third, we apply this property to the stock holdings of the hedger which eventually leads us to the conclusion that dynamic hedging produces positive feedback if the state-contingent payoff of the respective contingent claim is convex in the stock price.

Lemma 42 *The number of shares of the stock $\phi_n^1, n \in \{1, ..., N\}$, held during the time interval $[n-1, n[$ to dynamically hedge a contingent claim $A_N \in \mathbb{A}$ is,*

$$\phi_{n_j}^1 = \frac{A_{n_i} - A_{n_{i+1}}}{S_{n_i}^1 - S_{n_{i+1}}^1}, \tag{5.8}$$

where $i \in \{1, 3, ..., 2^n - 1\}$ and where $j = \frac{i+1}{2}$. $\phi_{n_j}^1$ is determined at the $j-th$ node at date $n-1$. A_{n_i} is the contingent claim's value at node i at date n.

Figure 5.4 illustrates the interdependencies of the different subscripts used in lemma 42. It should facilitate the understanding of the rather complex notation.

Proof. It is obvious that for a given date n,

$$\phi_n^0 \cdot (1+r)^n + \phi_n^1 \cdot S_n^1 = A_n, \tag{5.9}$$

must hold. Otherwise, the portfolio would not perfectly hedge the contingent claim. For given nodes i and $i+1, i \in \{1, 3, ..., 2^n - 1\}$, at date n, (5.9) gives rise to the linear system,

$$\left\{ \begin{array}{c} \phi_{n_j}^0 \cdot (1+r)^n + \phi_{n_j}^1 \cdot S_{n_i}^1 = A_{n_i} \\ \phi_{n_j}^0 \cdot (1+r)^n + \phi_{n_j}^1 \cdot S_{n_{i+1}}^1 = A_{n_{i+1}} \end{array} \right. .$$

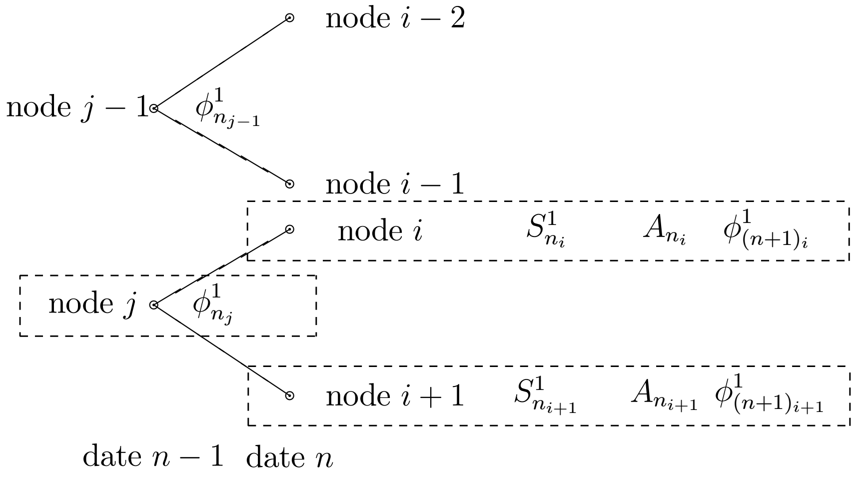

Figure 5.4: An extract from the tree.

Subtracting the second equation from the first one and rearranging terms yields the desired expression for $\phi^1_{n_j}$. ■

Heuristically, a positive feedback trader buys shares of the stock if its price rises and sells shares if its price declines. In our setup, positive feedback trading strategies can be defined formally as follows.

Definition 43 *Let a trading strategy* $(\phi_n)_{n\in\{0,\ldots,N\}}$ *be given. The trading strategy is a* ***positive feedback trading strategy*** *in the stock component* $(\phi^1_n)_{n\in\{0,\ldots,N\}}$ *if it satisfies,*

$$\forall (2 \le n \le N) : (\phi^1_n - \phi^1_{n-1}) \cdot (S^1_{n-1} - S^1_{n-2}) \ge 0. \tag{5.10}$$

To check whether a dynamic hedging strategy for a contingent claim with convex payoff produces positive feedback, i.e., whether it satisfies condition (5.10), we will draw on the following basic result for convex functions.

Lemma 44 *Let* $A : \mathbb{D} \to \mathbb{R}, S \mapsto A(S)$ *be a function. If* A *is convex in* S*, then it satisfies,*

$$\frac{A(S_1) - A(\overline{S})}{S_1 - \overline{S}} \ge \frac{A(\overline{S}) - A(S_2)}{\overline{S} - S_2},$$

where $S_1 > \overline{S} > S_2$.

Proof. Since A is assumed to be convex in S and since $S_1 > \overline{S} > S_2$, we have,

$$A(\overline{S}) \le \frac{\overline{S} - S_2}{S_1 - S_2} \cdot A(S_1) + \frac{S_1 - \overline{S}}{S_1 - S_2} \cdot A(S_2)$$

$$\begin{aligned}
&\Leftrightarrow (S_1 - S_2) \cdot A(\overline{S}) \leq \left(\overline{S} - S_2\right) \cdot A(S_1) + \left(S_1 - \overline{S}\right) \cdot A(S_2) \\
&\Leftrightarrow (S_1 - S_2) \cdot A(\overline{S}) + \overline{S} \cdot A(\overline{S}) \\
&\leq \overline{S} \cdot A(\overline{S}) + \left(\overline{S} - S_2\right) \cdot A(S_1) + \left(S_1 - \overline{S}\right) \cdot A(S_2) \\
&\Leftrightarrow \left(\overline{S} - S_2\right) \cdot A(\overline{S}) - \left(\overline{S} - S_2\right) \cdot A(S_1) \\
&\leq \left(S_1 - \overline{S}\right) \cdot A(S_2) - (S_1 - \overline{S}) \cdot A(\overline{S}) \\
&\Leftrightarrow \frac{A(\overline{S}) - A(S_1)}{S_1 - \overline{S}} \leq \frac{A(S_2) - A(\overline{S})}{S_2 - \overline{S}} \\
&\Leftrightarrow \frac{A(S_1) - A(\overline{S})}{S_1 - \overline{S}} \geq \frac{A(\overline{S}) - A(S_2)}{\overline{S} - S_2}.
\end{aligned}$$

This proves the lemma. ■

Eventually, we can establish the central result of positive feedback as proposition 45 below. It merely puts together the single pieces we have produced so far.

Proposition 45 *Let a contingent claim be given by,*

$$A_N : \Omega \to \mathbb{R}_+, \omega \mapsto A_N(S_N^1(\omega)).$$

The dynamic hedging strategy,

$$(\phi_n)_{n \in \{0,\dots,N\}},$$

for A_N is a positive feedback trading strategy in the stock component $(\phi_n^1)_{n \in \{0,\dots,N\}}$ if A_N is convex in S_N^1. Formally,

$$A_N \text{ is convex in } S_N^1 \Rightarrow (\phi_n^1)_{n \in \{0,\dots,N\}} \text{ satisfies } (5.10).$$

Proof. In what follows, we will work with an arbitrary seven-node sub-tree - starting at $n-2$ where $2 \leq n \leq N$ - of the whole tree. Figure 5.5 depicts such a sub-tree. In this figure, $k = \frac{j+1}{2}$, $j = \frac{i+1}{2}$ and consequently $k = \frac{i+3}{4}$.

The proof itself consists of three steps. First, we make sure that the price A_n of the contingent claim A_N at an arbitrary date n is convex in S_n^1. We need to do this first since it represents a prerequisite for applying lemma 44. Second and third, we prove by applying lemma 44 and a martingale argument, respectively, that condition (5.10) is satisfied under the convexity assumption.

Step 1: Proposition 31 and corollary 30 determine the price A_n of the contingent claim at date n as,

$$A_n = \beta_n^{-1} \cdot \mathbf{E}_n^{\mathbf{Q}} \left[\beta_N \cdot A_N(S_N^1)\right]$$

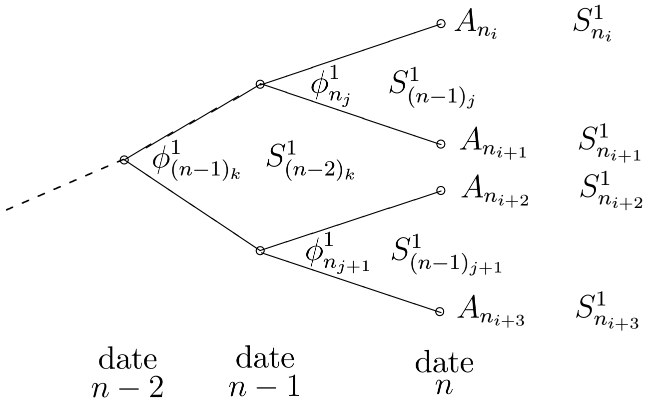

Figure 5.5: A seven-node sub-tree.

$$\begin{aligned}
&= \beta_n^{-1} \cdot \mathbf{E}_n^{\mathbf{Q}} \left[\beta_N \cdot A_N \left(S_n^1 \cdot \prod_{i=n+1}^{N} \frac{S_i^1}{S_{i-1}^1} \right) \right] \\
&= \beta_n^{-1} \cdot \beta_N \cdot \sum_{j=0}^{N-n} \left[\frac{(N-n)!}{(N-n-j)!j!} \cdot q^j \cdot (1-q)^{N-n-j} \right. \\
&\quad \left. \cdot A_N \left(S_n^1 \cdot (1+d)^j \cdot (1+u)^{N-n-j} \right) \right]. \qquad (5.11)
\end{aligned}$$

Recalling that $\beta_n^{-1}, \beta_N > 0$ and that A_N is convex in S_N^1 by assumption, we can deduce from (5.11) that A_n is a convex function in S_n^1.

Step 2: Consider now figure 5.5. Lemma 42 determines the necessary stock holdings during $[n-1, n[$ to hedge the contingent claim A_N as being,

$$\phi^1_{n_j} = \frac{A_{n_i} - A_{n_{i+1}}}{S^1_{n_i} - S^1_{n_{i+1}}} \text{ and} \qquad (5.12)$$

$$\phi^1_{n_{j+1}} = \frac{A_{n_{i+2}} - A_{n_{i+3}}}{S^1_{n_{i+2}} - S^1_{n_{i+3}}}. \qquad (5.13)$$

As before, $S^1_{n_i}$ denotes the stock price prevailing at node i at date n and $\phi^1_{n_j}$ denotes the stock position during $[n-1, n[$ as set up at j−th node at date $n-1$. Furthermore, it holds $A_{n_i} = \beta_n^{-1} \cdot \mathbf{E}^{\mathbf{Q}}_{n_i} \left[\beta_N \cdot A_N \left(S_N^1 \right) \right]$ with $\mathbf{E}^{\mathbf{Q}}_{n_i}[\cdot]$ denoting the conditional expectation given the information set $\mathcal{F}_n$ at node i at date n. After noting that,

$$S^1_{n_{i+1}} = S^1_{n_{i+2}} = S^1_{(n-2)_k} \cdot (1+u) \cdot (1+d),$$

and therewith $A^1_{n_{i+1}} = A^1_{n_{i+2}}$, one can apply lemma 44 to (5.12) and (5.13) to see that,

$$\phi^1_{n_j} \geq \phi^1_{n_{j+1}}.$$

Step 3: The discounted price process of the contingent claim, $(\beta_n A_n)_{n\in\{0,\ldots,N\}}$, is a martingale under $\mathbf{Q}$ which follows from proposition 1.2.3 in LAMBERTON and LAPEYRE (1996, 5). Given a node k at date $n-2$, we therefore have,

$$\begin{aligned} A_{(n-2)_k} &= (1+r)^{-1} \cdot \mathbf{E}^{\mathbf{Q}}_{(n-2)_k}\left[A_{(n-1)}\right] \\ &\Leftrightarrow (1+r)\cdot A_{(n-2)_k} = q\cdot A_{(n-1)_j} + (1-q)\cdot A_{(n-1)_{j+1}}, \end{aligned}$$

where $k = \frac{j+1}{2}$. And so,

$$\begin{aligned} & (1+r)\cdot\left(\phi^0_{(n-1)_k}\cdot S^0_{(n-2)_k} + \phi^1_{(n-1)_k}\cdot S^1_{(n-2)_k}\right) \\ = \;& q\cdot\left(\phi^0_{n_j}\cdot S^0_{(n-1)_j} + \phi^1_{n_j}\cdot S^1_{(n-1)_j}\right) \\ & +(1-q)\cdot\left(\phi^0_{n_{j+1}}\cdot S^0_{(n-1)_{j+1}} + \phi^1_{n_{j+1}}\cdot S^1_{(n-1)_{j+1}}\right) \\ \Leftrightarrow \;& \phi^0_{(n-1)_k}\cdot(1+r)^{n-1} + (1+r)\cdot\phi^1_{(n-1)k}\cdot S^1_{(n-2)_k} \\ = \;& \left[q\cdot\phi^0_{n_j} + (1-q)\cdot\phi^0_{n_{j+1}}\right]\cdot(1+r)^{n-1} \\ & + \left[q\cdot(1+u)\cdot\phi^1_{n_j} + (1-q)\cdot(1+d)\cdot\phi^1_{n_{j+1}}\right]\cdot S^1_{(n-2)_k}. \end{aligned}$$

Notice that $S^0_{(n-2)_k} \equiv (1+r)^{n-2}$. The last equality holds if both,

$$\phi^0_{(n-1)_k} = q\cdot\phi^0_{n_j} + (1-q)\cdot\phi^0_{n_{j+1}},$$

and,

$$(1+r)\cdot\phi^1_{(n-1)k} = q\cdot(1+u)\cdot\phi^1_{n_j} + (1-q)\cdot(1+d)\cdot\phi^1_{n_{j+1}}, \tag{5.14}$$

are satisfied. From (5.14),

$$\phi^1_{(n-1)k} = \underbrace{\frac{q\cdot(1+u)}{1+r}}_{\equiv\widetilde{q}}\cdot\phi^1_{n_j} + \underbrace{\frac{(1-q)\cdot(1+d)}{1+r}}_{\equiv\widetilde{q}'}\cdot\phi^1_{n_{j+1}}.$$

Recalling from the proof of lemma 37 that $q\cdot(1+u)+(1-q)\cdot(1+d) = 1+r$, one concludes that $\widetilde{q}+\widetilde{q}' = 1$ with $\widetilde{q},\widetilde{q}' > 0$. Recalling that $\phi^1_{n_j} \geq \phi^1_{n_{j+1}}$ (step 2), this implies,

$$\phi^1_{n_j} \geq \phi^1_{(n-1)k} \geq \phi^1_{n_{j+1}}. \tag{5.15}$$

Finally, noting that $S^1_{(n-1)_j} > S^1_{(n-2)_k} > S^1_{(n-1)_{j+1}}$ by construction and combining this with (5.15), one concludes that (5.10) is indeed satisfied because $n \geq 2$ was arbitrarily chosen. ■

5.4.2 A graphic illustration of the main result

The BLACK / SCHOLES option pricing formula allows insightful graphic demonstrations of the main result stated as proposition 45. Differentiating (5.5) with respect to the stock price, one obtains for European call options[10],

$$\frac{\partial C_t}{\partial S_t^1} = e^{-r^{BS} \cdot t^*} \cdot \Phi\left(d_1\right), \tag{5.16}$$

whereas one gets,

$$\frac{\partial P_t}{\partial S_t^1} = -e^{-r^{BS} \cdot t^*} \cdot \Phi\left(-d_1\right), \tag{5.17}$$

for European put options. d_1 is here as defined as in (5.6). The first derivative of the pricing formula with respect to the underlying stock price is commonly called the *delta* of the option. Roughly speaking, it represents the number of shares of the underlying stock contained in the date t hedge portfolio for an option according to BLACK and SCHOLES (1973). Hence, if delta increases with increasing stock price, i.e., if in the case of a European call option,

$$\frac{\partial^2 C_t}{\left(\partial S_t^1\right)^2} > 0,$$

then one clearly has a positive feedback hedging strategy.[11] The following example demonstrates graphically that this is actually true for European call and put options.

Consider a European option and the parameter specifications,

$$X = 50, r^{BS} = 0.05, \sigma^{BS} = 0.2,$$

where X denotes the exercise price, r^{BS} the risk-less interest rate, and σ^{BS} the BLACK / SCHOLES volatility parameter. Figure 5.6 plots the delta of a call version of the option against S_t^1 and t^*, figure 5.7 plots it for a put version in the same way. The delta of the call option is positive and only takes values between 0 and 1, whereas the delta of the put option is negative only taking values between -1 and 0. Though differing in their absolute values, both increase with a rising stock price, indicating positive feedback hedging in each case.

[10] Compare WILMOTT, HOWISON, and DEWYNNE (1995, 79-80).

[11] The second derivative of the pricing formula with respect to the underlying stock price is commonly called the *gamma* of the option. See chapter 14 of HULL (1997) for further details on the option 'Greeks' and their role in hedging options.

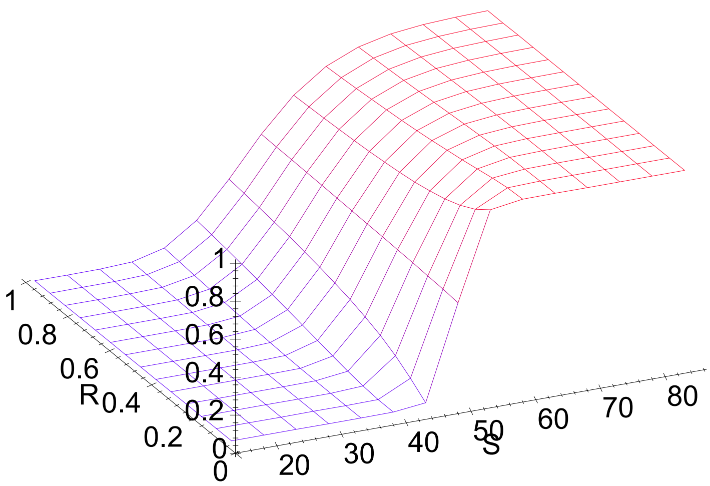

Figure 5.6: The delta of the European call option against S_t^1 (denoted by S) and t^* (denoted by R).

5.5 Summary

In this chapter, we have elaborated a crucial point about dynamic hedging strategies: It does not matter whether the hedged contingent claim is a European call option, a European put option or some other derivative. If the state-contingent payoff of the contingent claim is convex in the underlying's price, then the corresponding dynamic hedging strategy is a positive feedback strategy. Although the analysis in this chapter was mainly restricted to a discrete time setting, results from He (1990) imply that the positive feedback property of dynamic hedging strategies carries over to their continuous time counterparts. Furthermore, Sircar and Papanicolaou (1997), working in a generalized Black / Scholes setting in continuous time, validate the positive feedback result for convex payoffs from scratch.

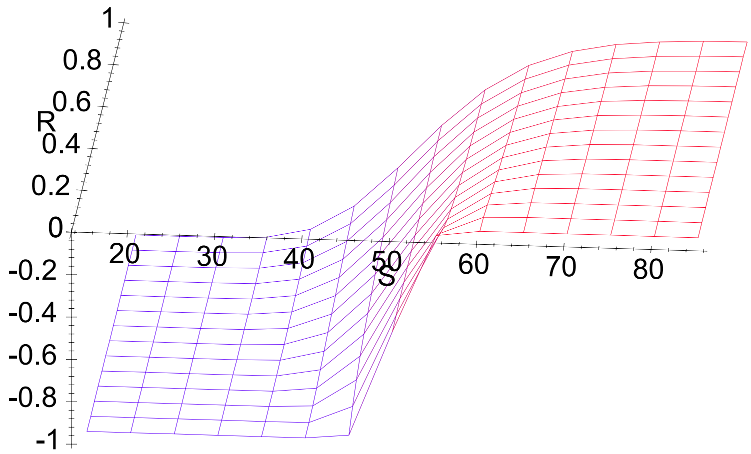

Figure 5.7: The delta of the European put option against S_t^1 (denoted by S) and t^* (denoted by R).

Chapter 6

Dynamic hedging and general equilibrium in complete markets

6.1 Introduction

This chapter poses a problem for an economy with a continuum of agents: Can positive feedback trading by agents with non-zero market weight[1] decrease stock price volatility if prices are set in equilibrium by risk-averse agents?

Inspired by empirical observations and experiences made by practitioners, many authors have suggested equilibrium models with hedgers having non-zero market weight for the purpose of exploring the potential impact of dynamic hedging on financial markets. As discussed in chapter 2, there are two approaches to analyze dynamic hedging in an equilibrium context: one stressing technicalities, such as positive feedback, the other stressing the role of information regarding the extent to which dynamic hedging takes place.

Almost all authors report that dynamic hedging *increases the volatility* of the underlying. Due to the overwhelming evidence, it seems likely that the observed effects have a common cause. Many authors have identified the payoff convexity of common derivatives as their preferred candidate. This is because payoff convexity causes the corresponding dynamic hedging strategy to pose positive feedback on markets. We formally verified this claim in chapter 5. Intuition suggests that such a trading behavior is likely to increase the volatility in those markets where it is implemented. The argument is that positive feedback trading pushes prices even higher after a price rise and drags them further down after a price fall. Thus, market volatility

[1] This is made precise in the body of the chapter.

increases. However, the work of BASAK (1995) constitutes a remarkable exception in this respect. His model predicts that market volatility decreases in the presence of portfolio insurance, although the given explanations seem a bit unsatisfactory. In particular, it is not clear what the crucial assumption is that leads to his results. BASAK (1995, 1081) himself admits:

> "Our result is in sharp contrast to the popular belief that portfolio insurance increases market volatility. The striking point about our conclusions is that this popular belief breaks down even in one of the most standard, best understood setups in finance [Lucas (1978) and CRRA preferences]."

The market model in which we embed our analysis in this chapter is, like the binomial model $\mathcal{M}^{CRR}$ of chapter 5, a special case of the market model $\mathcal{M}$ of chapter 4. However, there are several differences on which we want to comment briefly. The first is that we allow the set of agents $\mathbb{I}$ to include irrational or noise traders. In particular, there is a group of agents with non-zero market weight dynamically hedging contingent claims. The second main difference is that the security price process of the only risky security is determined in equilibrium rather than given exogenously. Two more minor differences are that we only consider the special case where $N = 2$ and that interest rates are implicit in the market model. A brief summary of the market model specifics follows.

The market model can be characterized as follows. The model economy is populated by a continuum of agents and lasts for the period $[0, 2]$. New information about the true state of the economy at the terminal date arrives at only three different dates $n \in \{0, 1, 2\}$. Two types of agents are active: hedgers implementing dynamic hedging programs for given contingent claims, and non-hedgers maximizing their expected utility from end of economy consumption. Agents can trade a stock and a bond. We prove that there exists a unique general equilibrium under our assumptions. In addition, the market model is complete under a common knowledge assumption so that the hedgers may achieve a perfect hedge at a given price.

The findings of BASAK (1995) let the author, referring to GROSSMAN and ZHOU (1996), reasonably conjecture the following:

> "In their numerical analysis, for a given set of exogenous parameters, they show that the net effect is an increase in market volatility. However, intuition would suggest that there might also be cases in which volatility would decrease." BASAK (1995, 1079).

In our simple model, it is possible to show that positive feedback hedging by agents with non-zero market weight is neither sufficient nor necessary to observe an increase in stock price volatility. In other words, we show by mainly relying on numerical examples, that negative feedback hedging can increase stock market volatility as well, and that positive feedback hedging may also decrease stock market volatility. Therefore, we can contribute to some extent to the resolution of the puzzle that arose through the findings of Basak (1995). Moreover, our findings represent evidence for the above quoted conjecture of Basak (1995) that the net volatility effect of dynamic hedging depends on the respective parameter specifications.

We are also able to derive strong analytical results for European call and put options. Volatility increases if European call or put options are hedged and it increases with increasing hedge demand. Moreover, hedge costs per option increase with increasing hedge demand, which is demonstrated for call options. Consequently, markets are no longer linear in the usual sense of the standard theory.[2]

The chapter proceeds as follows. Section 6.2 delineates the market model. Section 6.3 carries out the general equilibrium analysis and contains an existence and uniqueness proof. Section 6.4 explores some special cases. The comparative statics analysis of feedback effects from dynamic hedging takes place in section 6.5. In particular, we investigate dynamic hedging of European calls and puts in this section. Section 6.6 summarizes the main results. Finally, section 6.7 contains proofs of several results stated in the body of the chapter.

6.2 The market model

The market model is a special case of the market model,

$$\mathcal{M} = \{(\Omega, \wp(\Omega), \mathbb{F}, \mathbf{P}), N, \mathbb{S}, \mathbb{I}\},$$

presented in chapter 4. This section gives an overview of the relevant market model details.

6.2.1 Primitives

Consider an economy with uncertainty over the fixed time interval $[0, 2]$. News about the true state of the economy at the terminal date 2 arrives at dates $n \in \{0, 1, 2\}$ and all economic activity is observed at these dates. Uncertainty resolves according to a stochastic process, the so-called fundamental

[2] Compare definition 26, section 4.3.

state process $(\eta_n)_{n\in\{0,1,2\}}$ where $\forall n : \eta_n > 0$. The event tree corresponding to the state process has the following shape,

first date	intermediate date	terminal date
		η_2^{uu}
	η_1^u	
		η_2^{ud}
η_0		
		η_2^{du}
	η_1^d	
		η_2^{dd}
$n=0$	$n=1$	$n=2$

,

where we assume,

$$\eta_2^{uu} > \eta_2^{ud} \geq \eta_2^{du} > \eta_2^{dd}.$$

Four states ω of the economy are possible at the terminal date, while only two states are possible at the intermediate date. Formally, the state space is given by $\Omega = \{uu, ud, du, dd\}$. We call the node corresponding to η_1^u simply the 'u node' and that corresponding to η_1^d the 'd node' at date $n = 1$. The terminal nodes uu and ud may be reached from the u node only while the terminal nodes du and dd may only be reached from the d node.

The state process $(\eta_n)_{n\in\{0,1,2\}}$ generates the filtration $\mathbb{F} = (\mathcal{F}_n)_{n\in\{0,1,2\}}$ where,

$$\begin{aligned}\mathcal{F}_0 &= \{\varnothing, \Omega\},\\ \mathcal{F}_1 &= \{\varnothing, \{uu, ud\}, \{du, dd\}, \Omega\} \text{ and}\\ \mathcal{F}_2 &= \wp(\Omega).\end{aligned}$$

The probability measure $\mathbf{P}$ is strictly positive for all $\omega \in \Omega$, i.e., $\forall \omega \in \Omega$: $\mathbf{P}(\omega) > 0$. Ultimately, the filtered probability space $(\Omega, \wp(\Omega), \mathbb{F}, \mathbf{P})$ summarizes these pieces of information. To conclude, there is one *homogenous consumption good* available in the economy.

6.2.2 Securities

Two securities are traded at dates $n = 0$ and $n = 1$. One security is risk-less and called *bond*. It is in zero net supply. At $n = 2$, the bond pays out one unit of the homogenous good. Since it serves as the numeraire, its price is normalized to one giving rise to a bond price process $(S_n^0)_{n\in\{0,1,2\}}$ where $\forall n : S_n^0 \equiv 1$. By this normalization, interest rates are implicit in the model.

The other security is risky and called *stock*. The stock is in strictly positive supply. One share of the stock represents a claim to η_2 units of the homogenous good at $n = 2$. The stock price process $(S_n^1)_{n\in\{0,1,2\}}$ has therefore the same structure as the process $(\eta_n)_{n\in\{0,1,2\}}$,

first date	intermediate date	terminal date
		$S_2^1(uu) = \eta_2^{uu}$
	$S_1^1(u)$	
		$S_2^1(ud) = \eta_2^{ud}$
S_0^1		
		$S_2^1(du) = \eta_2^{du}$
	$S_1^1(d)$	
		$S_2^1(dd) = \eta_2^{dd}$
$n = 0$	$n = 1$	$n = 2$

Formally, the stock price process is adapted to the filtration $\mathbb{F}$, i.e., $\forall n : S_n^1$ is $\mathcal{F}_n-$measurable. Stock prices at $n = 0$ and $n = 1$ are set in equilibrium which will be examined later.

In summary, the sequence of events over time is,

$n = 0$	Markets open for both the stock and the bond Demand and supply enforce an equilibrium stock price S_0^1
$n = 1$	Stock and bond markets open again Demand and supply enforce an equilibrium stock price S_1^1
$n = 2$	One share of the stock pays η_2 units of the consumption good One unit of the bond pays 1 unit of the consumption good

Because of the assumption that prices at $n = 0$ and $n = 1$ are set in equilibrium, uniformed trading may influence the stock price at these dates. At $n = 2$, however, stock prices are completely determined by fundamentals so that uninformed trading cannot have any impact. As a result, these assumptions ensure that the model is consistent with empirical findings that uninformed trading is likely to influence stock prices in the short run as well as that stock price returns are mean-reverting in the long run.[3]

To conclude the description of securities markets, markets are assumed to function *perfectly* and information regarding relevant parameters is *complete*.

[3]Recall the discussion of the EMH and the noise trader approach in section 1.2. This section cites several empirical studies that support our statements.

Moreover, the market model is assumed to be *free of any arbitrage opportunity.*[4] If, as usual, $\mathbb{Q}$ denotes the set of all $\mathbf{P}$-equivalent probability measures that make the stock price process a martingale, Theorem 29 implies that $\mathbb{Q}$ will be non-empty in equilibrium.

6.2.3 Agents

The economy is populated by a continuum of agents $\mathbb{I} = [0, 1]$. All agents are endowed with one stock so that the stock is in constant aggregate supply of one. There are two types of agents, *hedgers* and *non-hedgers.* The proportion $\alpha \in [0, 1[$ of the hedgers is constant and so is the proportion of the non-hedgers, $1 - \alpha$. Expressions with respect to the hedgers are indicated by the superscript H, whereas expressions with respect to the non-hedgers are indicated by the superscript M. Whenever there is no danger of ambiguity, these superscripts are omitted. Among agents there is *perfect competition* (or *price taking*) as well as *complete* and *symmetric information.*[5] In particular, there is complete and symmetric information regarding all relevant market parameters, such as $\alpha, \mathbf{P}$, etc.

In the subsequent analysis, we will treat the groups of the hedgers and the non-hedgers as one single agent, respectively. Several assumptions are imposed that guarantee that this kind of aggregation is possible. The single representative agent of a group is then endowed with the aggregate endowment of the respective group. However, we will nonetheless keep on using the plural forms *hedgers* and *non-hedgers* to emphasize that there are many small agents and not one or two 'big' agents with the potential to influence markets.

Hedgers

Every hedger dynamically hedges a given attainable contingent claim $\overline{A}_2 \in \mathbb{A}$. Explanations *why* a hedger follows a dynamic hedging program lie beyond the scope of our analysis. However, one can observe in the marketplace that many market participants follow dynamic hedging programs. Imagine, for example, a bank or another financial institution selling customer tailored derivative securities over the counter (OTC). Although no liquid markets exist for custom tailored products in general, the institution can hedge against adverse market conditions. It can implement a dynamic hedging program to

[4] A description of these and other common model assumptions is found in section 1.1.

[5] Because of the assumption of perfect competition, the proportion α of the hedgers is to be interpreted as a *group* of agents rather than as a large agent with market weight α, as considered, for instance, in Jarrow (1992) and Jarrow (1994).

synthetically replicate the state-contingent payoff of the sold derivative securities. At maturity the proceeds of the synthetic securities and the obligation from the sold derivatives perfectly compensate each other. Throughout the analysis we should keep in mind the picture of such a financial institution.

Four assumptions are imposed for the hedgers:

H.1. *The aggregate state-contingent payoff* A_2 *that the hedgers actually realize by implementing their hedge programs satisfies,*

$$\eta_2 > A_2,$$

i.e., aggregate supply of the homogenous good at date $n = 2$ *suffices to cover the aggregate demand of the hedgers. All contingent claims considered are European, i.e., they have a payoff at* $n = 2$ *and no payoff before.*

H.2. *Hedgers can make a profit or loss which is denoted* $\pi = \alpha \cdot S_0^1 - A_0$. *Here*[6], $\pi \in]-(1-\alpha) \cdot S_0^1, \alpha \cdot S_0^1]$ *and* A_0 *denotes the* $n = 0$ *hedge costs associated with* A_2.

H.3. *There is no liquid market for the contingent claim* $\overline{A}_2$.

H.4. *There is complete and symmetric information about* $\overline{A}_2$.

Remark 9 *Assumption H.3. helps to focus on the main issue of the chapter: the impact of dynamic hedging on financial market equilibrium. The development of a consistent pricing theory, as done for example in* Jarrow (1994) *or* Frey (1996), *raises far-reaching questions that are not addressed here.*

The objective of the hedgers ($\equiv$ the representative hedger) is to ensure that they can honor their obligations from selling contingent claims of type $\overline{A}_2$, by implementing a dynamic hedge program. To achieve this, they choose the admissible trading strategy that minimizes the costs A_0 of dynamically

[6]The lower bound follows from,

$$\pi > - \underbrace{(1-\alpha) \cdot S_0^1}_{\text{intial endowment of the non-hedgers}},$$

the upper bound similarly from,

$$\pi \leq \underbrace{\alpha \cdot S_0^1}_{\text{initial endowment of the hedgers}}.$$

hedging, or more precisely, of super-replicating the aggregate target payoff $\widetilde{A}_2 \equiv \alpha \cdot \overline{A}_2$. Formally, they,

$$\min_{\phi \in \mathbb{T}} V_0(\phi)$$
$$\text{s.t.} V_2(\phi) \geq \widetilde{A}_2,$$

The costs are then given by $A_0 \equiv V_0(\phi)$. The actual payoff achieved by the hedgers is accordingly given as $A_2 \equiv V_2(\phi)$.

If the contingent claim $\overline{A}_2$ is attainable, this problem boils down to choosing $\phi \in \mathbb{T}$ such that $V_2(\phi) = \widetilde{A}_2$. Due to the absence of arbitrage, every trading strategy $\phi \in \mathbb{T}$ that generates $\widetilde{A}_2$ must yield the same hedge costs $A_0 \equiv V_0(\phi)$. In this case, the actual state-contingent payoff A_2 that the hedgers achieve coincides with the aggregate target payoff $\widetilde{A}_2$.

In summary, the hedgers seek to achieve a *complete hedge* in the sense that they end up with at least $\widetilde{A}_2(\omega)$ in all states $\omega \in \Omega$, i.e., $\forall \omega \in \Omega$: $A_2(\omega) \geq \widetilde{A}_2(\omega)$. They choose the admissible trading strategy that minimizes the associated hedge costs A_0. If the contingent claim $\overline{A}_2$, or equivalently $\widetilde{A}_2$, is even attainable, the hedgers achieve a *perfect hedge* meaning that $\forall \omega \in \Omega : A_2(\omega) = \widetilde{A}_2(\omega)$. The costs associated with the perfect hedge are then given by corollary 30 as $A_0 = \mathbf{E}_0^{\mathbf{Q}}[A_2]$ for all $\mathbf{P}-$equivalent martingale measures $\mathbf{Q} \in \mathbb{Q}$. In fact, we will see in the next section that the market model is complete so that every contingent claim is attainable. 'Real' super-replication will not become necessary until we analyze dynamic hedging in an incomplete markets environment in chapter 7. Therefore, chapter 7 seems to be the better place to discuss the concept of super-replication in greater detail.

Important for the plan in this chapter is that the state-contingent payoff A_2 that the hedgers actually achieve is independent of the equilibrium stock prices at dates $n = 0$ and $n = 1$. Since the hedgers solve their problem by backward induction, we can restrict our attention to their sub-problems at $n = 1$ to verify that A_2 is indeed independent of the stock prices at $n = 1$. Consider their problem, for example, at the u node,

$$\min_{\phi_2^0(u), \phi_2^1(u)} \quad \phi_2^0(u) + \phi_2^1(u) \cdot S_1^1(u)$$
$$\text{s.t.} \quad \phi_2^0(u) + \phi_2^1(u) \cdot S_2^1(uu) \geq \widetilde{A}_2(uu)$$
$$\phi_2^0(u) + \phi_2^1(u) \cdot S_2^1(ud) \geq \widetilde{A}_2(ud).$$

For a fixed cost level of $\widehat{A}_1$, the iso cost lines are given as,

$$\widehat{A}_1 = \phi_2^0(u) + \phi_2^1(u) \cdot S_1^1(u)$$
$$\Leftrightarrow \phi_2^0(u) = \widehat{A}_1 - \phi_2^1(u) \cdot S_1^1(u).$$

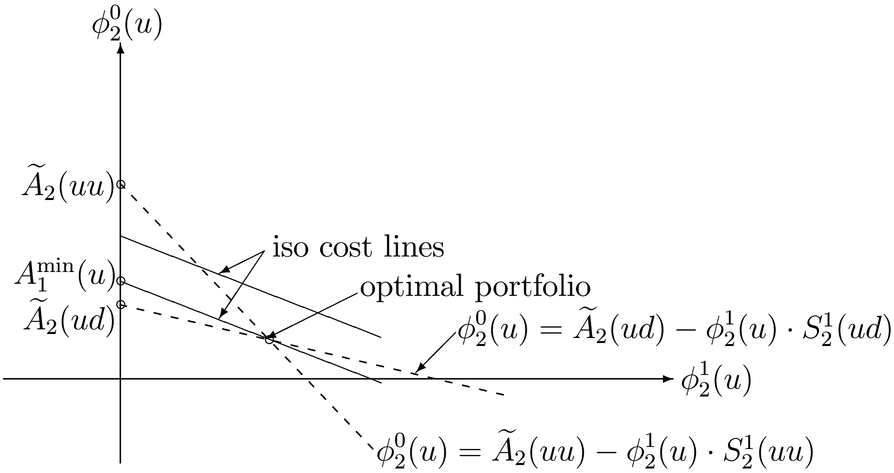

Figure 6.1: Determining the cost-minimizing super-replication portfolio.

Figure 6.1 illustrates the linear problem for the case where $\widetilde{A}_2(uu) > \widetilde{A}_2(ud)$. Note that one can conclude by arbitrage reasoning that,

$$S_1^1(u) \in]S_2^1(uu), S_2^1(ud)[.$$

If either $S_1^1(u) \geq S_2^1(uu)$ or $S_1^1(u) \leq S_2^1(uu)$, simple, risk-less arbitrage opportunities exist. In light of this, it is obvious why the cost-minimizing hedge portfolio is exclusively determined by the intersection of,

$$\phi_2^0(u) = \widetilde{A}_2(uu) - \phi_2^1(u) \cdot S_2^1(uu),$$

and,

$$\phi_2^0(u) = \widetilde{A}_2(ud) - \phi_2^1(u) \cdot S_2^1(ud).$$

This, in turn, shows that the actual achieved state-contingent payoff is independent of $S_1^1(u)$. A similar argument applies to the problem at the d node and contingent claims for which $\widetilde{A}_2(uu) \leq \widetilde{A}_2(ud)$ holds true.

So far so good, but what about the initial endowment $\alpha \cdot S_0^1$ of the hedgers? It can either be greater or smaller than the necessary initial investment A_0. Or - which is unlikely, but possible - it can be exactly the same. In a general equilibrium model one has to take this into account. By allowing the hedgers to make a profit or loss π, as made precise in assumption H.2. above, we ensure the consistency of the equilibrium model in this respect. What happens with π will be explained shortly. For the moment, to summarize matters, the sequence of actions and decisions made over time by the hedgers is,

$n = 0$	The hedgers construct a security portfolio that yields in combination with an appropriate trading strategy a $n = 2$ state-contingent payoff of at least $\widetilde{A}_2 \equiv \alpha \cdot \overline{A}_2$ The associated hedge costs A_0 satisfy $A_0 \lesseqgtr \alpha \cdot S_0^1$
$n = 1$	Security trading takes place The hedgers adjust their stock and bond positions
$n = 2$	They liquidate the portfolio: one share of the stock pays η_2 units of the consumption good, one unit of the bond pays 1 unit of the good, the hedgers obtain a sum of $A_2 \geq \widetilde{A}_2$

This chapter's focus lies on contingent claims whose payoff is convex. As has been shown in chapter 5, this class of contingent claims leads to positive feedback dynamic hedging strategies. The main result of chapter 5 is also applicable to the present market model.

Proposition 46 *Let the state-contingent payoff $\widetilde{A}_2 \equiv \alpha \cdot \overline{A}_2$ be attainable. The dynamic hedging strategy $(\phi_n)_{n\in\{0,1,2\}}$ associated with $\widetilde{A}_2$ is a positive feedback trading strategy in the stock component $(\phi_n^1)_{n\in\{0,1,2\}}$ if $\overline{A}_2$ is convex in S_2^1. Formally,*

$$\overline{A}_2 \text{ is convex in } S_2^1 \Rightarrow \phi_2^1(u) \geq \phi_1^1 \geq \phi_2^1(d).$$

$\phi_2^1(u)$ denotes the stock position set up by the hedgers at the u node at $n = 1$, $\phi_2^1(d)$ denotes the stock position set up at the d node at $n = 1$ and ϕ_1^1 finally denotes their stock position set up at $n = 0$.

Proof. The proposition follows from proposition 45 for $N = 2$. ■

Non-hedgers

Contrary to the hedgers, the non-hedgers act rational in the sense that they maximize their expected utility from consumption at the terminal date. For these agents, the following five assumptions are imposed throughout this and the next chapter:

M.1. *The non-hedgers derive utility from terminal date consumption only.*

M.2. *The non-hedgers' utility function,*

$$v : \mathbb{R}_+ \to \mathbb{R}, w \mapsto v(w),$$

is twice continuously differentiable. w denotes actual consumption at date $n = 2$.

M.3. *The non-hedgers bear the losses or obtain the profits of the hedgers so that their initial wealth is* $W_0^\pi \equiv (1-\alpha) \cdot S_0^1 + \pi$.

M.4. $v(\cdot)$ *is of hyperbolic absolute risk aversion (HARA) type.*

M.5. $v(\cdot)$ *satisfies* $v'(\cdot) > 0, v''(\cdot) < 0, \lim\limits_{w\to\infty} v'(w) = 0$ *and* $\lim\limits_{w\to 0} v'(w) = \infty$. $v'(\cdot)$ denotes the first derivative of $v(\cdot), v''(\cdot)$ the second derivative of $v(\cdot)$.

Remark 10 *A sensible interpretation of assumption M.3. is that the hedgers are firms owned by the non-hedgers. Assumption M.4. implies that we can aggregate among non-hedgers since these utility functions are known to generate linear* Engel *curves. The HARA class of utility functions, for instance, comprises those utility functions associated with CRRA or CARA. A detailed description of these functions may be found in* Milne (1979, 411) *or* Milne (1995)*. Assumption M.5. ensures that the non-hedgers are strictly risk-averse and that there are no corner solutions to the maximization problems. In fact, it further constrains the set of permissible utility functions.*

Since the hedgers act as automata in our model, non-hedgers set equilibrium prices according to their dynamic optimization problem. More precisely, equilibrium prices are set such that non-hedgers take security positions that make markets clear. The non-hedgers' problem ($\equiv$ the representative non-hedger's problem) takes on the form,

$$\max_{W_2 \in \mathbb{A}} \mathbf{E}_0^{\mathbf{P}}[v(W_2)] \tag{6.1}$$

$$\text{s.t. } W_2(uu) = \phi_2^0(u) + \phi_2^1(u) \cdot S_2^1(uu) \tag{6.2}$$

$$W_2(ud) = \phi_2^0(u) + \phi_2^1(u) \cdot S_2^1(ud) \tag{6.3}$$

$$W_2(du) = \phi_2^0(d) + \phi_2^1(d) \cdot S_2^1(du) \tag{6.4}$$

$$W_2(dd) = \phi_2^0(d) + \phi_2^1(d) \cdot S_2^1(dd) \tag{6.5}$$

$$\phi_1^0 + \phi_1^1 \cdot S_1^1(u) = \phi_2^0(u) + \phi_2^1(u) \cdot S_1^1(u) \tag{6.6}$$

$$\phi_1^0 + \phi_1^1 \cdot S_1^1(d) = \phi_2^0(d) + \phi_2^1(d) \cdot S_1^1(d) \tag{6.7}$$

$$(1-\alpha) \cdot S_0^1 + \pi = \phi_1^0 + \phi_1^1 \cdot S_0^1. \tag{6.8}$$

$W_2(\omega)$ and $S_2^1(\omega)$ denote the wealth and the stock price, respectively, in state $\omega \in \Omega$ at $n = 2$. $\phi_2^0(u)$ and $\phi_2^1(u)$ denote the bond position and stock position, respectively, at the u node at $n = 1$. Similarly, $\phi_2^0(d)$ and $\phi_2^1(d)$ denote these positions at the d node. ϕ_1^0 and ϕ_1^1 denote the bond and the stock position, respectively, at $n = 0$. $(1-\alpha) \cdot S_0^1$ is the initial endowment of the non-hedgers, expressed in units of the homogenous good, while π denotes the profits they receive from the hedgers or the losses of the hedgers they have to cover. Furthermore, (6.6) and (6.7) express that the trading strategy must be self-financing.

This problem can be condensed into,

$$\max_{\phi \in \mathbb{T}} \mathbf{E}_0^{\mathbf{P}}[v(V_2(\phi)] \tag{6.9}$$

$$\text{s.t. } V_0(\phi) = W_0^{\pi}, \tag{6.10}$$

where by definition $W_0^{\pi} \equiv (1-\alpha) \cdot S_0^1 + \pi$. This form of the problem is rather suggestive. Non-hedgers choose the admissible trading strategy that has initial costs of W_0^{π} and that maximizes their expected utility. Applying martingale methods finally yields the dynamic problem of the non-hedgers in the familiar static form,

$$\max_{W_2 \in \mathbb{A}} \mathbf{E}_0^{\mathbf{P}}[v(W_2)] \tag{6.11}$$

$$\text{s.t. } \forall \mathbf{Q} \in \mathbb{Q} : \mathbf{E}_0^{\mathbf{Q}}[W_2] = W_0^{\pi}. \tag{6.12}$$

Here, the budget constraint states that the expected value of the attainable state-contingent consumption payoff W_2 under any $\mathbf{P}-$equivalent martingale measure must equal the initial wealth of the non-hedgers.[7]

As an aside, note that we can also derive the problem of the non-hedgers in an unconstrained form. The following lemma will be used in section 6.4.

Lemma 47 *The problem of the non-hedgers (6.1)-(6.8) may also be expressed as the unconstrained problem,*

$$\arg\max_{\phi_1^1, \phi_2^1} \mathbf{E}_0^{\mathbf{P}}[v(W_0^{\pi} + \phi_1^1 \cdot \Delta S_1^1 + \phi_2^1 \cdot \Delta S_2^1)], \tag{6.13}$$

where $\phi_1^1 \in \mathbb{R}$, $\phi_2^1 \in \mathbb{R}^2$ *and* $\Delta S_n^1 \equiv (S_n^1 - S_{n-1}^1), n \in \{1, 2\}$.

Proof. Note that for any $\phi \in \mathbb{T}$,

$$V_2(\phi) - V_0(\phi) = G_2(\phi)$$
$$\Leftrightarrow V_2(\phi) = V_0(\phi) + G_2(\phi)$$

holds. Note further that the bond price does not change over time so that gains from trade can only be due to stock price changes ΔS_n^1, $n \in \{1, 2\}$.

[7] Notice that we used the strict monotonicity of the non-hedgers' utility function $v(\cdot)$ in all formulations of their problem. Strict monotonicity of $v(\cdot)$ particularly implies that the budget constraints are binding in each case.

In other words, the available $n = 2$ wealth of the non-hedgers equals their initial wealth plus gains from trading the stock,

$$\begin{aligned} & W_2 = W_0^{\pi} + G_2(\phi) \\ \Rightarrow \quad & W_2 = W_0^{\pi} + \phi_1^1 \cdot \Delta S_1^1 + \phi_2^1 \cdot \Delta S_2^1, \end{aligned} \tag{6.14}$$

where we used $W_0^{\pi} = V_0(\phi)$ and $V_2(\phi) = W_2$ [see problem (6.9) and (6.10)]. Consequently, the non-hedgers choose $\phi_1^1 \in \mathbb{R}$ and $\phi_2^1 \in \mathbb{R}^2$ such that they achieve a state-contingent payoff W_2 that maximizes their expected utility. As can be seen by comparing problems (6.9) / (6.10) and (6.11) / (6.12), choosing an admissible trading strategy is tantamount to choosing a state-contingent payoff. We obtain,

$$\begin{aligned} & \arg\max_{\phi \in \mathbb{T}} \mathbf{E}_0^{\mathbf{P}}[v(W_0^{\pi} + G_2(\phi)] \\ \Rightarrow & \arg\max_{\phi_1^1, \phi_2^1} \mathbf{E}_0^{\mathbf{P}}[v(W_0^{\pi} + \phi_1^1 \cdot \Delta S_1^1 + \phi_2^1 \cdot \Delta S_2^1)], \end{aligned}$$

as desired. See also, for instance, the discussion in section 5.1 of PLISKA (1997). ■

In summary, the sequence of decisions and actions of the non-hedgers is listed below.

$n = 0$	The non-hedgers invest into a security portfolio that, in combination with appropriate trading, maximizes their expected $n = 2$ utility
$n = 1$	Trading in the stock and the bond takes place The non-hedgers adjust their stock and bond positions
$n = 2$	To consume, they liquidate the security positions to obtain η_2 units of the homogenous good for each share of the stock and 1 unit of the homogenous good for each unit of the bond

Summary

To summarize matters presented in this section, we denote the present market model by,

$$\mathcal{M}^{cm} = \{(\Omega, \wp(\Omega), \mathbb{F}, \mathbf{P}), N = 2, \mathbb{S}^1, \mathbb{I}^{\alpha}\},$$

where,

- $\Omega = \{uu, ud, du, dd\}$,

- $\mathbb{F}$ is the filtration generated by the state process $(\eta_n)_{n\in\{0,1,2\}}$,
- $\mathbf{P}$ is strictly positive for all $\omega \in \Omega$,
- $N = 2$,
- $\mathbb{S}^1 = \{(S_n^k)_{n\in\{0,1,2\}} : k \in \{0,1\}\}$ where $\forall n : S_n^0 \equiv 1$ and
- $\mathbb{I}^\alpha = [0,1]$ with a proportion $\alpha \in [0,1[$ being hedgers and a proportion $1-\alpha$ being non-hedgers.

The section that follows will study general equilibrium in the market model $\mathcal{M}^{cm}$.

6.3 Equilibrium analysis

This section analyzes central issues related to general equilibrium in the case where both hedgers and non-hedgers are active in the market model $\mathcal{M}^{cm}$. Yet comparative static results are delegated to section 6.5.

Definition 48 *A* ***general equilibrium*** *for the market model,*

$$\mathcal{M}^{cm} = \{(\Omega, \wp(\Omega), \mathbb{F}, \mathbf{P}), N = 2, \mathbb{S}^1, \mathbb{I}^\alpha\},$$

is a collection of an equilibrium stock price process as well as trading strategies of the non-hedgers and hedgers,

$$\left\{ \begin{array}{l} (S_n^1)_{n\in\{0,1,2\}} \\ (\phi_n^M)_{n\in\{0,1,2\}} \\ (\phi_n^H)_{n\in\{0,1,2\}} \end{array} \right. , \tag{6.15}$$

such that,

- *non-hedgers reach their optimum,*
- *hedgers achieve their desired hedge, i.e., a state-contingent payoff of $A_2 \geq \widetilde{A}_2$ at $n = 2$, and*
- *markets clear,*

$$\phi_n^{0M} + \phi_n^{0H} = 0 \tag{6.16}$$
$$\phi_n^{1M} + \phi_n^{1H} = 1, \tag{6.17}$$

  for $n \in \{1,2\}$. ϕ_n^{0M} and ϕ_n^{0H} denote bond demand of the non-hedgers and the hedgers, respectively, while ϕ_n^{1M} and ϕ_n^{1H} denote stock demand of the non-hedgers and the hedgers, respectively.

It is clear that there can only exist a general equilibrium if there exists a solution to the non-hedgers maximization problem. This is due to the fact that the non-hedgers effectively set prices in equilibrium. In what follows, we want to establish existence and uniqueness of a general equilibrium. We determine the shape of the general equilibrium stock price process as well. This is mainly done by examining the problem of the non-hedgers. The existence and uniqueness results, accompanied by the shape of the equilibrium stock price, are found as Theorem 49 below. Afterwards, we derive the unique equilibrium equivalent martingale measure whereby we indirectly show market completeness.

Theorem 49 *There exists a unique general equilibrium in the market model* $\mathcal{M}^{cm} = \{(\Omega, \wp(\Omega), \mathbb{F}, \mathbf{P}), N = 2, \mathbb{S}^1, \mathbb{I}^\alpha\}$. *The equilibrium stock prices satisfy,*

$$S_n^1 = \frac{\mathbf{E}_n^{\mathbf{P}}\left[v'\left(\eta_2 - A_2\right) \cdot \eta_2\right]}{\mathbf{E}_n^{\mathbf{P}}\left[v'\left(\eta_2 - A_2\right)\right]}, \tag{6.18}$$

for $n \in \{0, 1\}$, *and* $S_2^1 = \eta_2$ *for* $n = 2$. $\mathbf{E}_n^{\mathbf{P}}$ *is the conditional expectation given the information set* $\mathcal{F}_n$, $v'(\cdot)$ *is the first derivative of the non-hedgers' utility function* $v(\cdot)$, η_2 *is the liquidating dividend of the stock at* $n = 2$ *and* A_2 *is the actual state-contingent payoff that the hedgers achieve at date* $n = 2$.

Proof. Sub-section 6.7.1 contains the proof. ■

Equation (6.18) will be central in the comparative statics analysis. It turns out to be a powerful tool in evaluating the impact of dynamic hedging on equilibrium prices in the market model $\mathcal{M}^{cm}$.

Proposition 50 characterizes the unique $\mathbf{P}-$equivalent martingale measure that prevails in equilibrium.

Proposition 50 *In equilibrium, the unique* $\mathbf{P}-$*equivalent martingale measure* $\mathbf{Q}^*$ *is determined by,*

$$\forall \omega \in \Omega : \mathbf{Q}^*(\omega) = \frac{\mathbf{P}(\omega) \cdot v'(\eta_2^\omega - A_2(\omega))}{\mathbf{E}_0^{\mathbf{P}}\left[v'(\eta_2 - A_2)\right]}. \tag{6.19}$$

η_2^ω *and* $A_2(\omega)$ *denote aggregate supply of the homogenous good and aggregate hedge demand in state* ω *at* $n = 2$, *respectively.*

Proof. See sub-section 6.7.2 for the proof. ■

We provide the shape of the state price density as a separate result.

Corollary 51 *The state price density for the economy takes on the form,*

$$\forall \omega \in \Omega : \mathbf{L}^*(\omega) = \frac{v'(\eta_2^\omega - A_2(\omega))}{\mathbf{E}_0^{\mathbf{P}}\left[v'(\eta_2 - A_2)\right]}.$$

Proof. The corollary follows immediately from the definition of the state price density, $\mathbf{L}^*(\omega) \equiv \frac{\mathbf{Q}^*(\omega)}{\mathbf{P}(\omega)}$. ■

In economic terms, an even more important corollary is the following.

Corollary 52 *The market model $\mathcal{M}^{cm} = \{(\Omega, \wp(\Omega), \mathbb{F}, \mathbf{P}), N = 2, \mathbb{S}^1, \mathbb{I}^\alpha\}$ is complete.*

Proof. Together propositions 35 and 50 imply market completeness. ■

Remark 11 *We shall emphasize that corollary 52 only applies to the market model $\mathcal{M}^{cm}$ because agents have, by assumption, complete and symmetric information about the equilibrium stock price process.*

We also get,

Lemma 53 *For $n \in \{0, 1\}$,*

$$\begin{aligned} S_n^1 &= \mathbf{E}_n^{\mathbf{Q}^*}[\eta_2] \\ &= \frac{\mathbf{E}_n^{\mathbf{P}}[\mathbf{L}^* \eta_2]}{\mathbf{E}_n^{\mathbf{P}}[\mathbf{L}^*]}. \end{aligned}$$

Proof. The lemma follows from Theorem 49, proposition 50 and corollary 51. ■

The situation where $\alpha = 0$ should serve as the benchmark case. Since this case is just a special case, one obtains from equation (6.18),

$$S_n^{1*} = \frac{\mathbf{E}_n^{\mathbf{P}}\left[v'(\eta_2) \cdot \eta_2\right]}{\mathbf{E}_n^{\mathbf{P}}\left[v'(\eta_2)\right]} = \mathbf{E}_n^{\mathbf{Q}^{**}}[\eta_2]. \tag{6.20}$$

Of course, $S_2^{1*} = \eta_2$ holds as before. The superscript "*" indicates the benchmark case and $\mathbf{Q}^{**}$ denotes the equilibrium martingale measure in that case. Bick (1987), Bick (1990) and He and Leland (1993) show that (6.20) is a necessary and sufficient condition for the stock price process $(S_n^{1*})_{n \in \{0,1,2\}}$ to be an equilibrium price process in a representative agent economy like our benchmark economy. The primary focus of their analyses, however, is on continuous time, complete markets settings.

The equilibrium stock prices at every single node of the binomial tree are labeled,

first date	intermediate date	terminal date
		$S_2^1(uu) = \eta_2^{uu}$
	$S_1^{1*}(u)$	
		$S_2^1(ud) = \eta_2^{ud}$
S_0^{1*}		
		$S_2^1(du) = \eta_2^{du}$
	$S_1^{1*}(d)$	
		$S_2^1(dd) = \eta_2^{dd}$
$n = 0$	$n = 1$	$n = 2$

.

The stock market clearing condition simplifies to,

$$\phi_n^{1M} = 1, \tag{6.21}$$

for $n \in \{1, 2\}$, with bond market equilibrium implied by WALRAS law if (6.21) holds. This particular kind of equilibrium is commonly called a 'no trade' equilibrium. Prices are set such that the non-hedgers have neither an incentive to buy nor to sell. Accordingly, bond demand equals zero. The resulting equilibrium trading strategy of the non-hedgers is simply their initial endowment, $\forall n \in \{0, 1, 2\} : (\phi_n) = (1, 0)$. Of course, $\pi = 0$ for $\alpha = 0$.

We are now readily equipped to analyze the impact of dynamic hedging on financial market equilibrium. However, before starting the analysis of feedback effects, it seems helpful to stress the slightly unusual definition of volatility used in this and the next chapter.

Definition 54 *The* ***volatility*** σ *of the stock price is defined as the difference between the two possible* $n = 1$ *equilibrium stock prices, i.e.,* $\sigma \equiv S_1^1(u) - S_1^1(d)$ *for the economy with hedgers and* $\sigma^* \equiv S_1^{1*}(u) - S_1^{1*}(d)$ *for the benchmark economy.*

Although this notion of volatility is not common, it lends itself in the context of a two period binomial model. It is, for instance, the same notion of volatility as in GROSSMAN (1988).

6.4 Some special cases

In this section, we explore a parameterized version of the market model $\mathcal{M}^{cm} = \{(\Omega, \wp(\Omega), \mathbb{F}, \mathbf{P}), N = 2, \mathbb{S}^1, \mathbb{I}^\alpha\}$ to highlight several aspects of dynamic hedging in a framework like ours. Purposely, this section focuses on contingent claims that are, in a sense, borderline cases.[8] Numerical computations for more realistic contingent claims are delegated to section 7.6 of the next chapter.

A popular belief is that positive feedback trading increases volatility in imperfectly liquid markets. This belief is supported by numerous studies as laid out in chapter 2. In similar vein, some argue that negative feedback trading decreases volatility in such markets. However, without further qualification these statements are mere conjectures and do not generally hold. We are able to construct examples in the market model $\mathcal{M}^{cm}$ with which we can refute both conjectures.[9] In particular, we show in this section, taking as reference the volatility σ^* in the benchmark case $\alpha = 0$, the following:

- Dynamic hedging of a contingent claim with convex payoff leading to *positive feedback* trading may both,
  - *increase* (sub-section 6.4.2) and
  - *decrease* (sub-section 6.4.3)

  the volatility σ of the stock price.
- Dynamic hedging of a contingent claim with non-convex payoff leading to *negative feedback* trading may *increase* the volatility σ of the stock price (sub-section 6.4.4).

6.4.1 The example economy

Consider the market model $\mathcal{M}^{cm}$ with the following parameter specifications:

$$\begin{aligned}\eta_2 &\in \{12.1, 10, 10, 8.264\},\\ \forall\omega &\in \Omega : \mathbf{P}(\omega) = 0.25 \text{ and}\\ v &: \mathbb{R}_+ \to \mathbb{R}, w \mapsto 1 - e^{-\frac{1}{2}\cdot w}.\end{aligned}$$

The non-hedgers exhibit CARA of $\frac{1}{2}$ which can be easily checked by applying the definition of absolute risk aversion (definition 13). For illustration purposes only, figure 6.2 depicts the utility function and its first and second derivative.[10]

[8] We should be a bit more precise and point out that our findings are robust for several classes of contingent claims with non-zero measure. In other words, we do not consider extreme cases with measure zero.

[9] What we actually do is showing that the two conjectures fail to hold generically.

[10] The current setting departs from the assumption that $\lim_{w\to 0} v'(w) = \infty$ since the assumption is not necessary for our purposes here. However, because the exponential utility function belongs to the HARA class of utility functions, we can simply aggregate among non-hedgers.

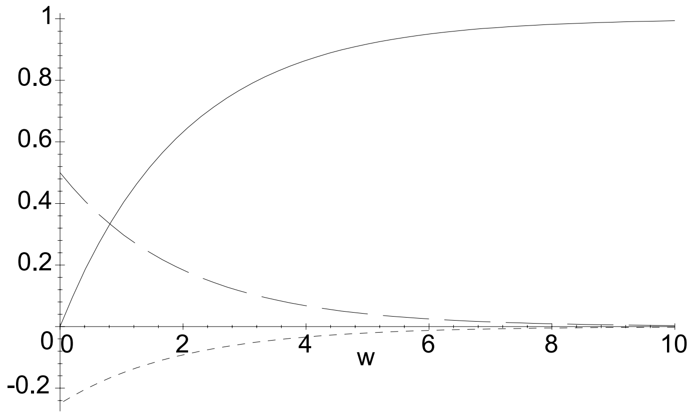

Figure 6.2: The utility function $v(w) = 1 - e^{-\frac{1}{2}w}$ [solid line], its first derivative $v'(w)$ [dashed line] and its second derivative $v''(w)$ [dotted line].

Using (6.20), one calculates for the stock prices in the benchmark case $\alpha = 0$,

first date	intermediate date	terminal date
		$S_2^1(uu) = 12.1$
	$S_1^{1*}(u) = 10.5444$	
		$S_2^1(ud) = 10$
$S_0^{1*} = 9.2814$	$\sigma^* = 1.7671$	
		$S_2^1(du) = 10$
	$S_1^{1*}(d) = 8.7773$	
		$S_2^1(dd) = 8.264$
$n = 0$	$n = 1$	$n = 2$

.

Stock price volatility, measured as the difference between the two possible stock prices at date $n = 1$, amounts to $\sigma^* = 1.7671$.

6.4.2 Positive feedback increases volatility

The first case of dynamic hedging we examine is the dynamic hedging of a European call option. This call option is given by,

$$C_2 : \Omega \to \mathbb{R}_+, \omega \mapsto \begin{cases} 3 \text{ if } \omega = uu \\ 1 \text{ if } \omega = ud \\ 1 \text{ if } \omega = du \\ 0 \text{ if } \omega = dd \end{cases}.$$

Assuming that the hedgers' market weight is 10% ($\alpha = 0.1$), the equilibrium stock prices, as determined by (6.18), are now,

first date	intermediate date	terminal date
		$S_2^1(uu) = 12.1$
	$S_1^1(u) = 10.5856$	
		$S_2^1(ud) = 10$
$S_0^1 = 9.3291$	$\sigma = 1.7901$	
		$S_2^1(du) = 10$
	$S_1^1(d) = 8.7955$	
		$S_2^1(dd) = 8.264$
$n = 0$	$n = 1$	$n = 2$

Volatility increases from $\sigma^* = 1.7671$ to $\sigma = 1.7901$ or by 1.3016%.

To get a clearer picture of the forces that lead to these observations, we introduce another approach to derive equilibrium stock prices. The subsequent analysis is based on considerations regarding the non-hedgers' stock demand. In particular, we use the fact that the stock demand of the non-hedgers must equal aggregate stock supply adjusted for the stock holdings of the hedgers. Recall that we made similar considerations in a partial equilibrium context in section 1.3. Formally, we work with the stock market clearing condition (6.17). Doing so enables us to provide intuitive graphic illustrations of how dynamic hedging perturbs the equilibrium price process.

From lemma 47, the stock demand functions of the non-hedgers at both the u and the d node at $n = 1$ can be derived in explicit form. At $n = 1$, the non-hedgers face the problem,

$$\arg\max_{\phi_2^1} \mathbf{E}_1^{\mathbf{P}}[v(W_1 + \phi_2^1 \cdot \Delta S_2^1)]. \tag{6.22}$$

At $n = 1, W_1$ is given since the optimal investment $\widetilde{\phi}_1^1$ in the stock done at

$n = 0$ cannot be revised at this date. Given $\widetilde{\phi}_1^1$, $W_1 = W_0^\pi + \widetilde{\phi}_1^1 \cdot \Delta S_1$.[11]

In light of (6.22), the first order condition for optimality is,

$$\mathbf{E}_1^{\mathbf{P}}[v'(W_1 + \phi_2^1 \cdot \Delta S_2^1) \cdot \Delta S_2^1] \stackrel{!}{=} 0. \tag{6.23}$$

Depending on whether the actual node at $n = 1$ is u or d, (6.23) implicitly defines the stock demand function $\phi(S)$ of the non-hedgers at the respective node. We conduct the necessary calculations to derive the explicit stock demand function for the u node only. Taking into account the parameter specifications, one obtains from (6.23),

$$\begin{aligned}
&\frac{1}{2} \cdot \left(\frac{1}{2} \cdot e^{-\frac{1}{2}\cdot(W_1+\phi\cdot(12.1-S))} \cdot (12.1 - S)\right) \\
&+\frac{1}{2} \cdot \left(\frac{1}{2} \cdot e^{-\frac{1}{2}\cdot(W_1+\phi\cdot(10-S))} \cdot (10 - S)\right) = 0 \\
&\Rightarrow \ln\left[e^{-\frac{1}{2}\cdot(W_1+\phi\cdot(12.1-S))} \cdot (12.1 - S)\right] \\
&= \ln\left[-e^{-\frac{1}{2}\cdot(W_1+\phi\cdot(10-S))} \cdot (10 - S)\right] \\
&\Leftrightarrow \ln\left[-\frac{e^{-\frac{1}{2}\cdot(W_1+\phi\cdot(12.1-S))}}{e^{-\frac{1}{2}\cdot(W_1+\phi\cdot(10-S))}} \cdot \frac{(12.1 - S)}{(10 - S)}\right] = 0 \\
&\Leftrightarrow \ln\left[-e^{-\frac{2.1}{2}\cdot\phi} \cdot -\frac{(12.1 - S)}{(S - 10)}\right] = 0 \\
&\Leftrightarrow -\frac{2.1}{2} \cdot \phi = -\ln\left[\frac{(12.1 - S)}{(S - 10)}\right] \\
&\Leftrightarrow \phi = \frac{2}{2.1} \cdot \ln\left[\frac{(12.1 - S)}{(S - 10)}\right].
\end{aligned} \tag{u node}$$

As we see, the stock demand function $\phi(S)$ is independent of the actual wealth W_1 at $n = 1$. This is a well-known result for CARA preferences.[12] Similar calculations to those for the u node yield for the stock demand function at the d node,

$$\phi = \frac{2}{1.736} \cdot \ln\left[\frac{(10 - S)}{(S - 8.264)}\right]. \tag{d node}$$

[11] Here, we benefit from the fact that expected utility maximizing agents act dynamically consistent.

[12] Refer, for instance, to Eichberger and Harper (1997, 25-29).

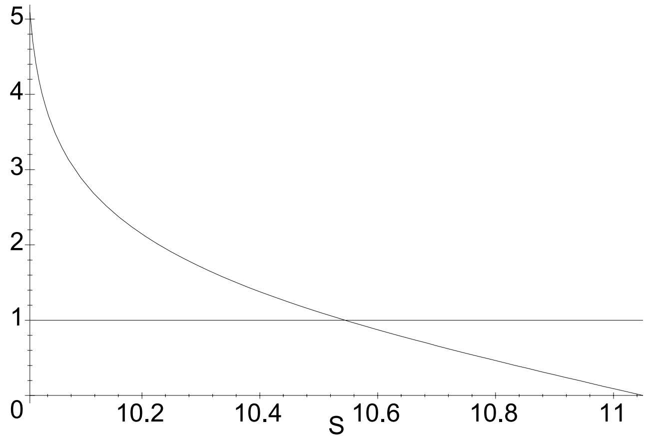

Figure 6.3: Stock market equilibrium in the example at the u node at $n = 1$ for $\alpha = 0$.

Stock market clearing in the benchmark case prevails if (6.21) is satisfied. In particular, this translates into,

$$u \text{ node} \quad : \quad \frac{2}{2.1} \cdot \ln \left[\frac{(12.1 - S)}{(S - 10)} \right] = 1, \tag{6.24}$$

$$d \text{ node} \quad : \quad \frac{2}{1.736} \cdot \ln \left[\frac{(10 - S)}{(S - 8.264)} \right] = 1. \tag{6.25}$$

From these conditions, one computes for the equilibrium stock prices at $n = 1$ $S_1^{1*}(u) = 10.5444$ and $S_1^{1*}(d) = 8.7773$, respectively. These values are, as desired, the same as those derived via the pricing equation (6.20). Conditions (6.24) and (6.25) are illustrated in figures 6.3 and 6.4, respectively. Since the stock demand functions of the non-hedgers are strictly decreasing (over the relevant range) and satisfy 'nice' limit properties, both equilibria are unique.

Similarly, the impact of dynamic hedging can be evaluated with the help of the stock market clearing condition (6.17). To do so, however, one has to calculate the stock demand of the hedgers first. The usual arbitrage argument for complete markets yields,

$$\begin{aligned} \phi_2^{1H}(u) &= \alpha \cdot \frac{C_2(uu) - C_2(ud)}{\eta_2^{uu} - \eta_2^{ud}} \\ &= 0.1 \cdot \frac{3 - 1}{12.1 - 10} \\ &= 0.095238, \end{aligned} \tag{6.26}$$

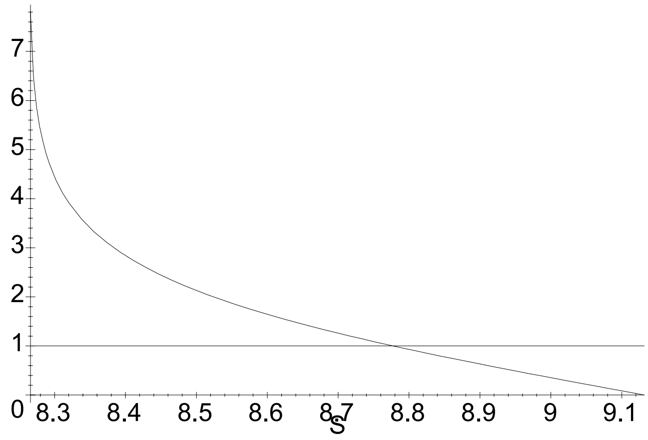

Figure 6.4: Stock market equilibrium in the example at the d node at $n = 1$ for $\alpha = 0$.

and,

$$\begin{aligned} \phi_2^{1H}(d) &= \alpha \cdot \frac{C_2(du) - C_2(dd)}{\eta_2^{du} - \eta_2^{dd}} \\ &= 0.1 \cdot \frac{1 - 0}{10 - 8.264} \\ &= 0.05760. \end{aligned} \tag{6.27}$$

From condition (6.17),

$$\begin{aligned} \phi_2^{1M} + \phi_2^{1H} &= 1 \\ \Leftrightarrow \phi_2^{1M} &= 1 - \phi_2^{1H}. \end{aligned} \tag{6.28}$$

At the u node, for example, the aggregate stock supply 'from the point of view of the non-hedgers', i.e., the right hand side of (6.28), is $1 - \phi_2^{1H}(u) = 0.9048$. Using the stock demand function of the non-hedgers as derived above, the stock market clearing condition (6.28) gives,

$$\begin{aligned} u \text{ node} &: \quad \frac{2}{2.1} \cdot \ln\left[\frac{(12.1 - S)}{(S - 10)}\right] = 0.9048, \\ d \text{ node} &: \quad \frac{2}{1.736} \cdot \ln\left[\frac{(10 - S)}{(S - 8.264)}\right] = 0.9424. \end{aligned}$$

These conditions determine $S_1^1(u) = 10.5856$ and $S_1^1(d) = 8.7955$ as the $n = 1$ equilibrium stock prices. This is in line with the results obtained when relying on (6.18).

Applying the same method, one can evaluate the impact of dynamically hedging a European put option defined as follows,

$$P_2 : \Omega \to \mathbb{R}_+, \omega \mapsto \begin{cases} 0 \text{ if } \omega = uu \\ 1 \text{ if } \omega = ud \\ 1 \text{ if } \omega = du \\ 3 \text{ if } \omega = dd \end{cases}.$$

For $\alpha = 0.1$, the stock positions of the hedgers at $n = 1$ are $\phi_2^{1H}(u) = -0.047619$ and $\phi_2^{1H}(d) = -0.11521$, respectively. With everything else unchanged, the $n = 1$ equilibrium stock prices are easily computed to be $S_1^1(u) = 10.5245$ and $S_1^1(d) = 8.7419$. They yield a stock price volatility of $\sigma = 1.7826$, which corresponds to a rise in volatility of 0.8771% relative to the benchmark case.

6.4.3 Positive feedback decreases volatility

In contrast to a wide-spread belief, dynamic hedging of contingent claims with convex payoffs can also decrease the underlying's volatility. We can construct a contingent claim that has a convex payoff and that nevertheless decreases the volatility of the stock in the parameterized market model $\mathcal{M}^{cm}$. Consider, for instance, the contingent claim,

$$K_2 : \Omega \to \mathbb{R}_+, \omega \mapsto \begin{cases} 3.9 \text{ if } \omega = uu \\ 4 \quad \text{ if } \omega = ud \\ 4 \quad \text{ if } \omega = du \\ 4.1 \text{ if } \omega = dd \end{cases}.$$

The necessary stock holdings of the hedgers to perfectly hedge the contingent claim are $\phi_2^{1H}(u) = -0.0047619$ and $\phi_2^{1H}(d) = -0.0057604$ when $\alpha = 0.1$. The stock market clearing condition (6.28) then implies that,

$$\begin{aligned} u \text{ node} \quad &: \quad \frac{2}{2.1} \cdot \ln \left[\frac{(12.1 - S)}{(S - 10)} \right] = 1 + 0.0047619, \\ d \text{ node} \quad &: \quad \frac{2}{1.736} \cdot \ln \left[\frac{(10 - S)}{(S - 8.264)} \right] = 1 + 0.0057604, \end{aligned}$$

must hold in equilibrium. These conditions in turn determine the $n = 1$ equilibrium stock prices as being $S_1^1(u) = 10.5424$ and $S_1^1(d) = 8.7755$. The stock price volatility is now $\sigma = 1.7669$, which is slightly less than the stock price volatility in the benchmark economy, $\sigma^* = 1.7671$. The remainder of this sub-section is devoted to the explanation of this striking observation.

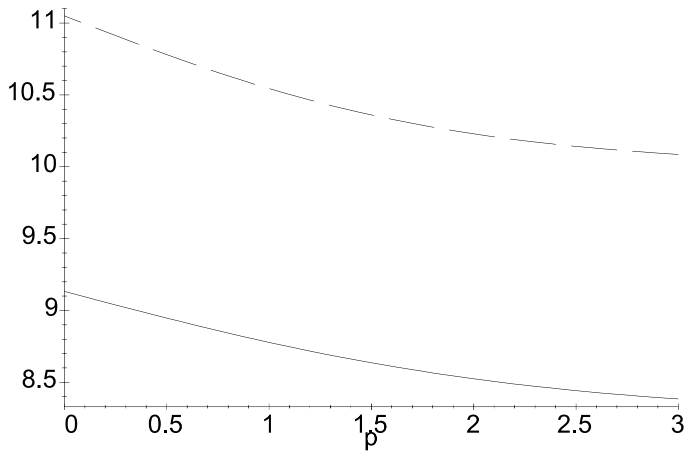

Figure 6.5: The stock demand functions $\phi(S)$ of the non-hedgers. The dashed line corresponds to the u node, the solid line to the d node. In this figure, ϕ is denoted by p.

As pointed out in section 1.3, in a setting like the present one, we can interpret the *elasticity* of the non-hedgers' stock demand function as a measure for the liquidity of the stock market. In what follows, we argue that the observed effect in regard to the convex payoff K_2 can be explained by differences in the stock market liquidity. Recall that the hedgers sell short $\phi_2^{1H}(u) = -0.0047619$ shares of the stock at the u node to hedge the payoff K_2. This leads to a stock price fall of 0.002 from 10.5444 to 10.5424. At the d node, the hedgers sell short $\phi_2^{1H}(d) = -0.0057604$ shares of the stock which causes the stock to drop by 0.0018 from 8.7773 to 8.7755. Even though the hedgers sell more shares of the stock short at the d node than at the u node, $\phi_2^{1H}(d) < \phi_2^{1H}(u)$, the impact on the stock price is stronger at the u node. Apparently, the stock market is more liquid at the d node than at the u node. To verify this, we can draw on the elasticity of the non-hedgers' stock demand function. The elasticity ϵ of their stock demand function around 1 at the u node is,

$$\epsilon(u) = \left| \frac{\frac{\Delta\phi}{\phi}}{\frac{\Delta S_1^1(u)}{S_1^1(u)}} \right| = \frac{\frac{0.00476}{1}}{\frac{0.002}{10.544}} = 25.09,$$

whereas it is,

$$\epsilon(d) = \left| \frac{\frac{\Delta\phi}{\phi}}{\frac{\Delta S_1^1(d)}{S_1^1(d)}} \right| = \frac{\frac{0.0057604}{1}}{\frac{0.0018}{8.7773}} = 28.09,$$

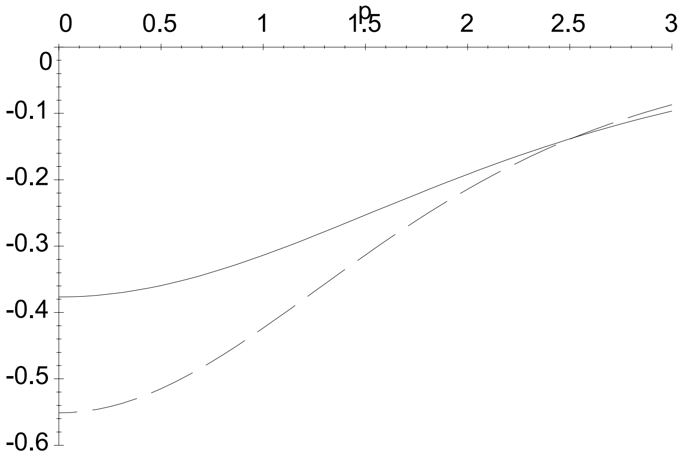

Figure 6.6: The first derivatives of the stock demand functions of the non-hedgers at the u node [dashed line] and at the d node [solid line]. In this figure, p denotes ϕ.

at the d node. Indeed, $\epsilon(u) < \epsilon(d)$, indicating that the stock market is actually more liquid at the d node than at the u node for quantities around 1.

As we see in the market model $\mathcal{M}^{cm}$, an important determinant of the impact of dynamic hedging on equilibrium stock prices and the stock price volatility is the liquidity of the stock market. Furthermore, it comes true that a measure suited to describe the liquidity of the stock market in the market model $\mathcal{M}^{cm}$ is the elasticity of the non-hedgers' demand function.

Since we are concerned with the impact of dynamic hedging on the stock price only and do not need to compare different markets with each other, we can also use a simpler measure as indicator for stock market liquidity. Such a measure is the *slope* of the stock demand functions. In order to present the two stock demand functions at $n = 1$ in a single diagram which makes it possible to compare them directly, figure 6.5 displays the stock demand functions of the non-hedgers in a diagram where ϕ is put on the horizontal axis and S is put on the vertical axis. Figure 6.6 depicts the first derivatives of these functions. A brief inspection reveals that the stock demand function at the u node is steeper around 1 than the stock demand function at the d node. This explains why in the case of the convex payoff K_2 volatility decreases. The smaller shock caused by the hedgers at the u node,

$$\left|\phi_2^{1H}(u)\right| < \left|\phi_2^{1H}(d)\right|,$$

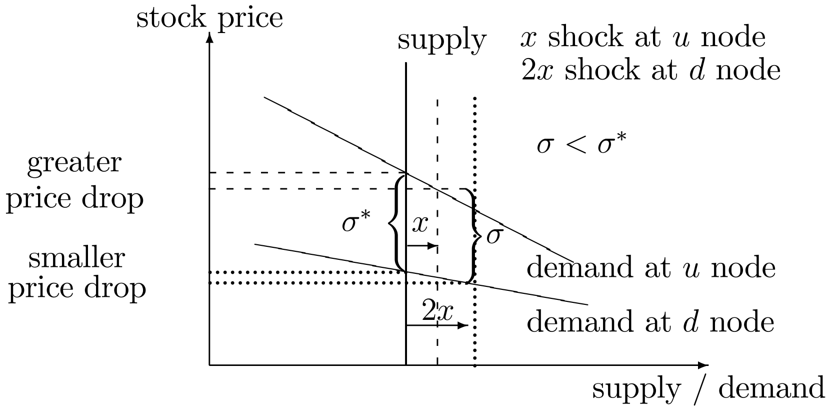

Figure 6.7: Positive feedback hedging decreases volatility.

nevertheless causes a bigger price drop. This is due to the stock demand function being steeper around 1 at the u node than at the d node. Figure 6.7 summarizes these statements in a single, schematic diagram.

6.4.4 Negative feedback increases volatility

The last contingent claim we consider in this section is,

$$NK_2 : \Omega \to \mathbb{R}_+, \omega \mapsto \begin{cases} 4.1 \text{ if } \omega = uu \\ 4 \quad \text{if } \omega = ud \\ 4 \quad \text{if } \omega = du \\ 3.9 \text{ if } \omega = dd \end{cases}.$$

This contingent claim has a non-convex payoff. For $\alpha = 0.1$, one can easily check that $\phi_2^{1H}(u) = 0.0047619$ and $\phi_2^{1H}(d) = 0.0057604$. Thus, there is negative feedback. One computes for the stock market equilibrium prices $S_1^1(u) = 10.5464$ and $S_1^1(d) = 8.7791$ yielding a stock price volatility of $\sigma = 1.7673$. As a result, σ slightly surpasses the volatility $\sigma^* = 1.7671$ in the benchmark economy. We see that negative feedback hedging may increase the volatility of the stock price, thereby contradicting results of Balduzzi, Bertola, and Foresi (1995). The argument to explain this observation parallels the one for the case of the contingent claim K_2 where volatility decreased. As a matter of fact, the contingent claims K_2 and NK_2 are, in a sense, put and call versions of the same contingent claim.

Summary

Table 6.1 summarizes some of the main results derived in this section. In the table, volatility in the benchmark case is normalized to 100%.

Table 6.1: Summary of results.

state \ *contingent claim*	benchmark	C_2	P_2	K_2	NK_2
uu	0	3	0	3.9	4.1
ud	0	1	1	4	4
du	0	1	1	4	4
dd	0	0	3	4.1	3.9
volatility [%]	100	101.30	100.88	99.99	100.01

Considering all these findings, it is now clear that payoff convexity, or equivalently positive feedback trading, is neither sufficient nor necessary in the market model $\mathcal{M}^{cm}$ to observe an increase in stock price volatility caused by dynamic hedging. To put it the other way round, positive feedback dynamic hedging may both increase and decrease stock price volatility. Similarly, negative feedback hedging may also both increase and decrease stock price volatility where we omitted demonstrating the second part of this assertion.

6.5 Comparative statics analysis

In this section, we consider more general cases of contingent claims. We particularly investigate typical European call and put options. To begin with, assume that $\alpha \in]0,1[$ and that every hedger hedges one option out of a set of permissible options. Suppose that this set contains I European call options and J European put options. These options have different strike prices but the same expiration $n = 2$. They satisfy,

$$\begin{aligned} \forall i &\in \{1, ..., I\} : \\ C_2^i &: \mathbb{R}_{++} \to \mathbb{R}_+, S_2^1 \mapsto \max\{S_2^1 - K_i, 0\}, \quad (6.29) \\ \eta_2^{uu} &> K_i > \eta_2^{ud}, \quad (6.30) \end{aligned}$$

in the case of the calls and,

$$\begin{aligned} \forall j &\in \{1, ..., J\} : \\ P_2^j &: \mathbb{R}_{++} \to \mathbb{R}_+, S_2^1 \mapsto \max\{X_j - S_2^1, 0\}, \quad (6.31) \\ \eta_2^{du} &> X_j > \eta_2^{dd}, \quad (6.32) \end{aligned}$$

in the case of the puts. Using $S_2^1 = \eta_2$, (6.30) implies for the state-contingent payoff of the $i-$th call with strike K_i,

$$C_2^i = \begin{cases} \eta_2^{uu} - K_i & \text{if } \omega = uu \\ 0 & \text{if } \omega = ud \\ 0 & \text{if } \omega = du \\ 0 & \text{if } \omega = dd \end{cases}.$$

Accordingly, (6.32) implies for the state-contingent payoff of the $j-$th put with strike X_j,

$$P_2^j = \begin{cases} 0 & \text{if } \omega = uu \\ 0 & \text{if } \omega = ud \\ 0 & \text{if } \omega = du \\ X_j - \eta_2^{dd} & \text{if } \omega = dd \end{cases}.$$

Suppose now that a fraction $\rho \in [0,1]$ of α hedges call options and that the remaining fraction $1-\rho$ hedges put options. The analysis may be simplified by the use of, what we call, *average strike options.*[13]

On the one hand, if $\alpha_i \in [0,\rho]$ represents the proportion of hedgers hedging call option $i \in \{1,...,I\}$, we must have $\sum_{i=1}^I \alpha_i = \rho \cdot \alpha$. There exists an average strike call option $\overline{C}_2$ that satisfies[14],

$$\sum_{i=1}^{I} \alpha_i \cdot C_2^i = \rho \cdot \alpha \cdot \overline{C}_2,$$

with,

$$\overline{C}_2 \quad : \quad \mathbb{R}_{++} \rightarrow \mathbb{R}_+, S_2^1 \mapsto \max\{S_2^1 - \overline{K}, 0\}, \tag{6.33}$$
$$\eta_2^{uu} \quad > \quad \overline{K} > \eta_2^{ud}. \tag{6.34}$$

On the other hand, if $\alpha_j \in [0, 1-\rho]$, with $\sum_{j=1}^J \alpha_j = (1-\rho)\cdot\alpha$, is the fraction of the hedgers who hedge put option $j \in \{1,...,J\}$, then there exists an average strike put $\overline{P}_2$ satisfying,

$$\sum_{j=1}^{J} \alpha_j \cdot P_2^j = (1-\rho) \cdot \alpha \cdot \overline{P}_2,$$

[13]Not to be confused with exotic options whose strike price depends on the average stock price during a certain period of time.

[14]Simply define,

$$\overline{C}_2 \equiv \frac{\sum_{i=1}^I \alpha_i \cdot C_2^i}{\rho \cdot \alpha}.$$

with,

$$\begin{aligned} \overline{P}_2 &: \mathbb{R}_{++} \to \mathbb{R}_+, S_2^1 \mapsto \max\{\overline{X} - S_2^1, 0\}, & (6.35) \\ \eta_2^{ud} &> \overline{X} > \eta_2^{dd}. & (6.36) \end{aligned}$$

After all, we end up with a situation that we can manage conveniently. As a matter of fact, a picture for the hedge activity in the economy emerges that can be characterized in terms of the average strike options only. In particular, the general case of the $I + J$ different options is equivalent to a situation where,

- a proportion ρ of the hedgers, i.e., a proportion $\rho \cdot \alpha$ of the whole population, dynamically hedges call options $\overline{C}_2$ and where,
- the remaining proportion $1 - \rho$ of the hedgers, amounting to a proportion $(1-\rho)\cdot\alpha$ of the whole population, dynamically hedges put options $\overline{P}_2$.

For the comparative statics analysis, it seems helpful to consider two cases, an extreme one and a general one,

(a) $\rho = 1$ and

(b) $\rho \in [0, 1]$.

Both cases will be examined separately in sub-sections 6.5.1 and 6.5.2.

6.5.1 Dynamic hedging of calls

Consider first case (a), i.e., that all hedgers hedge call options of type $\overline{C}_2$. Importantly, market completeness implies that $\overline{C}_2$ is attainable in any case. As the comparative statics analysis below reveals, dynamic hedging of such call options causes the $n = 0$ price of the underlying stock to climb. The intuition behind this is that a hedger has to take a long position in the underlying to hedge the call option. This additional demand for the underlying stock causes the price of the stock price to increase. Moreover, the observed effect is stronger the higher the market weight α is of the hedgers.

Proposition 55 *In the market model $\mathcal{M}^{cm}$, the higher the market weight α of the hedgers is who dynamically hedge calls of type $\overline{C}_2$, the higher is the $n = 0$ equilibrium stock price S_0^1.*

Proof. The proof is delegated to sub-section 6.7.3. ■

An immediate consequence of proposition 55 is,

Corollary 56 *The stock price S_0^1 is higher in the presence of hedgers dynamically hedging calls of type $\overline{C}_2$ than in their absence, $S_0^1 > S_0^{1*}$.*

Proof. Clear. ■

Similarly, the additional stock demand by the hedgers at the u node at $n = 1$ causes the stock price $S_1^1(u)$ to climb. It also turns out that dynamic hedging of European calls leads to an increase in the stock price volatility, which is more amplified the higher the aggregate hedge demand is. But first we have,

Proposition 57 *In the market model $\mathcal{M}^{cm}$, the $n = 1$ equilibrium stock price $S_1^1(u)$ at the u node increases with increasing market weight α of hedgers dynamically hedging calls of type $\overline{C}_2$.*

Proof. Sub-section 6.7.4 contains the proof. ■

An immediate corollary of proposition 57 is the increase in volatility.

Corollary 58 *The volatility σ of the stock price increases with increasing market weight α of hedgers dynamically hedging calls of type $\overline{C}_2$.*

Proof. It suffices to realize that according to (6.18), $\frac{\partial}{\partial\alpha}S_1^1(d) = 0$, and therefore, $\frac{\partial}{\partial\alpha}\sigma = \frac{\partial}{\partial\alpha}(S_1^1(u) - S_1^1(d)) > 0$. ■

We also obtain,

Corollary 59 *The volatility σ of the stock price is higher in the presence of hedgers with non-zero market weight α dynamically hedging calls of type $\overline{C}_2$ than in their absence, $\sigma > \sigma^*$.*

Proof. From $\frac{\partial S_1^1(u)}{\partial\alpha} > 0$ and the observation that according to (6.18), $S_1^1(d) = S_1^{1*}(d)$, follows,

$$\begin{aligned} \sigma &= S_1^1(u) - S_1^1(d) \\ &= S_1^1(u) - S_1^{1*}(d) \\ &> S_1^{1*}(u) - S_1^{1*}(d) \\ &= \sigma^*, \end{aligned}$$

since $S_1^1(u) > S_1^{1*}(u)$. ■

We can now draw on our knowledge about the unique, $\mathbf{P}-$equivalent martingale measure $\mathbf{Q}^*$ to show that the hedge costs per call option $\overline{C}_2$ are dependent on the market weight α of hedgers. More precisely, hedge costs per call option rise with rising α.

Proposition 60 *In the market model $\mathcal{M}^{cm}$, hedge costs $\overline{C}_0$ per call option contract $\overline{C}_2$ as given by,*

$$\overline{C}_0 = \mathbf{E}_0^{\mathbf{Q}^*}[\overline{C}_2],$$

increase with increasing market weight α of hedgers dynamically hedging calls of type $\overline{C}_2$. $\mathbf{Q}^*$ denotes the unique equilibrium $\mathbf{P}-$equivalent martingale measure associated with a certain α. Formally,

$$\frac{\partial \overline{C}_0}{\partial \alpha} > 0.$$

Proof. We know that in equilibrium the unique $\mathbf{P}-$equivalent martingale measure is provided in (6.19). Consider first the three states of the world where the call option $\overline{C}_2$ expires worthless, i.e., the states $\omega \in \{ud, du, dd\}$. The martingale probabilities for these states are given as,

$$\mathbf{Q}^*(\omega) = \frac{\mathbf{P}(\omega) \cdot v'(\eta_2^{\omega})}{\mathbf{E}_0^{\mathbf{P}}[v'(\eta_2 - A_2)]},$$

where,

$$\begin{aligned}
\mathbf{E}_0^{\mathbf{P}}[v'(\eta_2 - A_2)] &= \mathbf{P}(uu) \cdot v'\left(\eta_2^{uu} - \alpha \cdot \left(\eta_2^{uu} - \overline{K}\right)\right) + \mathbf{P}(ud) \cdot v'(\eta_2^{ud}) \\
&\quad + \mathbf{P}(du) \cdot v'(\eta_2^{du}) + \mathbf{P}(dd) \cdot v'(\eta_2^{dd}).
\end{aligned}$$

Since $\frac{\partial \mathbf{E}_0^{\mathbf{P}}[v'(\eta_2 - A_2)]}{\partial \alpha} > 0$, we deduce that for $\omega \in \{ud, du, dd\} : \frac{\partial \mathbf{Q}^*(\omega)}{\partial \alpha} < 0$. This, in turn, implies that $\frac{\partial \mathbf{Q}^*(uu)}{\partial \alpha} > 0$ because of $\mathbf{Q}^*$ being a probability measure satisfying $\sum_{\omega \in \Omega} \mathbf{Q}^*(\omega) = 1$.

In equilibrium, the hedge costs $\overline{C}_0$ of the call option $\overline{C}_2$ satisfy,

$$\begin{aligned}
\overline{C}_0 &= \mathbf{E}_0^{\mathbf{Q}^*}\left[\overline{C}_2\right] \\
&= \mathbf{Q}^*(uu) \cdot \left(\eta_2^{uu} - \overline{K}\right).
\end{aligned}$$

Combining this with $\frac{\partial \mathbf{Q}^*(uu)}{\partial \alpha} > 0$ yields the assertion that the hedge costs for a call option of type $\overline{C}_2$ rise if the market weight α of the hedgers rises. ■

This sub-section focused on call options only. Similar results can be obtained for the case where both call and put options are hedged, which is demonstrated in the next sub-section.

6.5.2 Dynamic hedging of calls and puts

This sub-section explores case (b) where $\rho \cdot \alpha$ hedgers hedge calls of type $\overline{C}_2$ and $(1 - \rho) \cdot \alpha$ hedgers hedge puts of type $\overline{P}_2$. The aggregate payoff A_2 that is to be hedged is $A_2 \equiv \alpha \cdot \overline{A}_2$ where,

$$\begin{aligned}
\overline{A}_2 &\equiv \left[\rho \cdot \begin{pmatrix} \eta_2^{uu} - \overline{K} \\ 0 \\ 0 \\ 0 \end{pmatrix} + (1-\rho) \cdot \begin{pmatrix} 0 \\ 0 \\ 0 \\ \overline{X} - \eta_2^{dd} \end{pmatrix} \right] \\
&= \begin{pmatrix} \rho \cdot \left(\eta_2^{uu} - \overline{K}\right) \\ 0 \\ 0 \\ (1-\rho) \cdot \left(\overline{X} - \eta_2^{dd}\right) \end{pmatrix}. \qquad (6.37)
\end{aligned}$$

Market completeness implies that the hedgers can indeed achieve this state-contingent payoff. These preliminary considerations which effectively ensure that we can treat the group of hedgers as if every hedger would hedge the same contingent claim $\overline{A}_2$, now lead to,

Proposition 61 *In the market model $\mathcal{M}^{cm}$, regardless of the actual value of $\rho \in [0,1]$, the $n = 1$ equilibrium stock price $S_1^1(u)$ at the u node is non-decreasing and the $n = 1$ equilibrium stock price $S_1^1(d)$ at the d node is non-increasing in the proportion α of hedgers dynamically hedging calls of type $\overline{C}_2$ and puts of type $\overline{P}_2$. For $\rho > 0$, $S_1^1(u)$ is strictly increasing in α. For $\rho < 1$, $S_1^1(d)$ is strictly decreasing in α.*

Proof. See sub-section 6.7.5. ■

The intuition behind proposition 61 is that hedgers aiming at hedging call options must buy shares of the stock at the u node at $n = 1$. Since they simultaneously seek to hedge put options, they must sell shares of the stock short at the d node at $n = 1$. For the non-hedgers to sell shares of the stock to the hedgers at the u node, the stock price must rise. Similarly, the stock price must fall at the d node to motivate the non-hedgers to absorb the additional supply. As a result, volatility increases,

Corollary 62 *Regardless of the actual value of $\rho \in [0,1]$, the volatility σ of the stock price increases with increasing market weight α of hedgers dynamically hedging calls $\overline{C}_2$ and puts $\overline{P}_2$.*

Proof. Immediate consequence of proposition 61 and the definition of volatility σ. ■

As an important sub-case of the last corollary, one has,

Corollary 63 *The volatility σ of the stock price is higher in the presence of hedgers with non-zero market weight α dynamically hedging calls of type $\overline{C}_2$ and puts of type $\overline{P}_2$ than in their absence, or equivalently, $\sigma > \sigma^*$.*

Proof. Clear. ■

This finishes the comparative statics analysis. As seen, we were able to verify that dynamic hedging of certain classes of contingent claims inevitably increases the volatility of the stock price in the market model $\mathcal{M}^{cm}$. The present parameter setting includes the interesting case $\rho = 0$. In this case, hedgers only hedge put options which can be interpreted as a portfolio insurance economy, as considered in Brennan and Schwartz (1989), Basak (1995) or Grossman and Zhou (1996). Our findings with regard to the particular type of put option $\overline{P}_2$ are consistent with those of Brennan and Schwartz (1989) and Grossman and Zhou (1996), while they clearly contradict those of Basak (1995).

The classes of contingent claims for which our results are unambiguous are quite special. In particular, the contingent claims analyzed in this section have a strictly positive payoff in only one state. If we allowed for more general payoff structures, the results would be ambiguous.[15] Therefore, in our model, we must come to the conclusion that positive feedback dynamic hedging, even if it is induced by rather typical contingent claims, does not necessarily increase market volatility in imperfectly liquid markets. This is in sharp contrast to the overwhelming evidence delineated in chapter 2.

6.6 Summary

In this chapter, we analyzed dynamic hedging in a general equilibrium framework. We proved that under the assumptions of complete and symmetric information a unique general equilibrium exists in the market model $\mathcal{M}^{cm}$. Furthermore, these assumptions ensured that the market model is complete.

Our findings indicate that dynamic hedging has the potential to influence security prices considerably. We found that dynamic hedging of standard European call and put options lets the stock price volatility climb in the market model $\mathcal{M}^{cm}$. Dynamic hedging of such options leads to positive feedback trading so that it is intuitively appealing that it increases volatility. Our result can also be considered rather robust insofar as we worked with widely accepted utility functions and did not specify any other market model parameter.

The potential of dynamic hedging to perturb the stock price process stems from the general equilibrium approach. Such an approach implies imperfectly liquid security markets. As a result, hedgers' trading activity has an impact on the stock prices set by the risk-averse non-hedgers in equilibrium. In

[15] In fact, our results would break down if we allowed contingent claims with strictly positive payoffs in states other than uu and dd. In chapter 7, this will become evident.

contrast, this never happens if the stock price process is given exogenously, such as in standard contingent claim pricing models.

It has often been conjectured that positive feedback trading generally increases volatility in imperfectly liquid markets. At first sight and also in light of our results regarding certain call and put options, this sounds reasonable. Yet examples demonstrated that positive feedback trading may both increase and decrease the volatility of the stock price. In a similar fashion, another example revealed that negative feedback trading may increase the volatility of the stock price.

It seems fair to conclude that, without further qualification, positive feedback trading is neither sufficient nor necessary to observe an increase in volatility in a general equilibrium setting. Therefore, our results may contribute to the resolution of the puzzle that arose when Basak (1995) found that positive feedback trading decreases volatility in his model. Since our examples were based on a parameter setting where the non-hedgers had CARA preferences, we can exclude that the crucial assumption leading to Basak's (1995) findings is to be seen in the CRRA preferences. Moreover, our results clearly support a conjecture formulated by Basak (1995) and quoted in the introduction of this chapter that the net volatility effect of dynamic hedging crucially depends on the specific market model parameters.

6.7 Mathematical proofs

6.7.1 Proof of Theorem 5.4

The proof comprises four steps:

1. We show that there exists a solution to the non-hedgers' problem.
2. We verify that the solution is unique.
3. We derive the unique solution.
4. After all, we bring into play an equilibrium condition to verify the shape of the equilibrium stock prices and to show that the found equilibrium is unique as well.

Steps 3 and 4 are necessary for both date $n = 0$ and date $n = 1$.

Step 1: To begin with, recall that the problem of the non-hedgers is to,

$$\max_{W_2 \in \mathbb{A}} \mathbf{E}_0^{\mathbf{P}}[v(W_2)] \tag{6.38}$$

$$\text{s.t. } \forall \mathbf{Q} \in \mathbb{Q} : \mathbf{E}_0^{\mathbf{Q}}[W_2] = W_0^{\pi}. \tag{6.39}$$

No arbitrage and Theorem 29 imply the existence of a solution to this problem. We nonetheless want to sketch a proof. Uniqueness of the solution is established thereafter.
No arbitrage and Theorem 29 imply that the set $\mathbb{Q}$ of $\mathbf{P}-$equivalent martingale measures is non-empty. As a result, the budget set as given in (6.39) is bounded and closed, i.e., it is compact. Mathematically, compactness of the budget set follows since $\forall \omega \in \Omega, \forall \mathbf{Q} \in \mathbb{Q} : \mathbf{Q}(\omega) > 0$. This is ensured by the requirement that all martingale measures $\mathbf{Q} \in \mathbb{Q}$ be $\mathbf{P}-$equivalent. For a fixed $\widehat{\mathbf{Q}} \in \mathbb{Q}$ and $W_0^\pi < \infty$, (6.39) becomes,

$$\begin{gathered}\widehat{\mathbf{Q}}(uu) \cdot W_2(uu) + \widehat{\mathbf{Q}}(ud) \cdot W_2(ud) \\ +\widehat{\mathbf{Q}}(du) \cdot W_2(du) + \widehat{\mathbf{Q}}(dd) \cdot W_2(dd) = W_0^\pi,\end{gathered}$$

which should illustrate the argument. Now note that the utility function $v(\cdot)$ and the conditional expectation $\mathbf{E}_0^{\mathbf{P}}[\cdot]$ are both continuous. As a consequence, $\mathbf{E}_0^{\mathbf{P}}[v(\cdot)]$ is continuous as well. Since the objective function is continuous and the budget set is compact, the WEIERSTRASS Theorem implies the existence of a solution to problem (6.38) and (6.39). See also chapter 3 in SUNDARAM (1996).

Step 2: Next we turn to the uniqueness of the solution. Since the utility function $v(\cdot)$ is strictly concave, $\mathbf{E}_0^{\mathbf{P}}[v(\cdot)]$ is strictly quasi-concave which implies according to proposition 2.10 in KREPS (1990) strictly convex preferences. Strictly convex preferences, in turn, imply according to proposition 2.11 in KREPS (1990) uniqueness of the solution to problem (6.38) and (6.39). Finally, the assumptions $\lim_{w\to\infty} v'(w) = 0$ and $\lim_{w\to 0} v'(w) = \infty$ ensure that $W_2 \in \mathbb{R}^4_{++}$.

Step 3: To derive the optimal solution to the non-hedgers' problem, it is helpful to work with a martingale basis instead of the whole set $\mathbb{Q}$. Therefore, let $\mathbb{Q}^B \equiv \{\mathbf{Q}_1^B, \mathbf{Q}_2^B, ..., \mathbf{Q}_J^B\}$ form a martingale basis. As before, $\mathbf{P}(\omega)$ and $\mathbf{Q}_j^B(\omega)$ denote the probability for state ω to unfold under the probability measure $\mathbf{P}$ and $\mathbf{Q}_j^B$, respectively. Rewrite the problem of the non-hedgers as,

$$\max_{\phi_2^0,\phi_2^1 \in \mathbb{R}^2} \mathbf{E}_0^{\mathbf{P}}[v(\phi_2^0 + \phi_2^1 \cdot S_2^1)] \tag{6.40}$$

$$\text{s.t. } \forall \mathbf{Q}_j^B \in \mathbb{Q}^B : \mathbf{E}_0^{\mathbf{Q}_j^B}[\phi_2^0 + \phi_2^1 \cdot S_2^1] = W_0^\pi. \tag{6.41}$$

Here we substituted for $W_2 = \phi_2^0 + \phi_2^1 \cdot S_2^1$. In view of the upcoming need for partial derivatives and the fact that $\phi_2^0, \phi_2^1 \in \mathbb{R}^2$, it seems worthwhile to write problem (6.40) and (6.41) in full detail,

$$\begin{aligned}\max_{\phi_2^0(u),\phi_2^0(d),\phi_2^1(u),\phi_2^1(d)} \quad & \mathbf{P}(uu)\cdot v(\phi_2^0(u)+\phi_2^1(u)\cdot S_2^1(uu)) \\ & +\mathbf{P}(ud)\cdot v(\phi_2^0(u)+\phi_2^1(u)\cdot S_2^1(ud)) \\ & +\mathbf{P}(du)\cdot v(\phi_2^0(d)+\phi_2^1(d)\cdot S_2^1(du)) \\ & +\mathbf{P}(dd)\cdot v(\phi_2^0(d)+\phi_2^1(d)\cdot S_2^1(dd)) \\ & \text{s.t. } \forall \mathbf{Q}_j^B \in \mathbb{Q}^B : \\ & \mathbf{Q}_j^B(uu)\cdot(\phi_2^0(u)+\phi_2^1(u)\cdot S_2^1(uu)) \\ & +\mathbf{Q}_j^B(ud)\cdot(\phi_2^0(u)+\phi_2^1(u)\cdot S_2^1(ud)) \\ & +\mathbf{Q}_j^B(du)\cdot(\phi_2^0(d)+\phi_2^1(d)\cdot S_2^1(du)) \\ & +\mathbf{Q}_j^B(dd)\cdot(\phi_2^0(d)+\phi_2^1(d)\cdot S_2^1(dd)) = W_0^\pi .\end{aligned}$$

One obtains the LAGRANGIAN function,

$$\begin{aligned}\mathcal{L}(\phi_2^0,\phi_2^1,\lambda) &= \mathbf{E}_0^{\mathbf{P}}\left[v(\phi_2^0+\phi_2^1\cdot S_2^1)\right] \\ &+\sum_{j=1}^{J}\lambda_j\cdot\left(W_0^\pi-\mathbf{E}_0^{\mathbf{Q}_j^B}[\phi_2^0+\phi_2^1\cdot S_2^1]\right).\end{aligned}$$

The first order conditions, which are both necessary and sufficient here are,

$$\left\{\begin{aligned}\frac{\partial\mathcal{L}}{\partial\phi_2^1(u)} &= \mathbf{P}(uu)\cdot v'(\phi_2^0(u)+\phi_2^1(u)\cdot S_2^1(uu))\cdot S_2^1(uu) \\ &+\mathbf{P}(ud)\cdot v'(\phi_2^0(u)+\phi_2^1(u)\cdot S_2^1(ud))\cdot S_2^1(ud) \\ &-\textstyle\sum_{j=1}^J\lambda_j\cdot\left[\mathbf{Q}_j^B(uu)\cdot S_2^1(uu)+\mathbf{Q}_j^B(ud)\cdot S_2^1(ud)\right]\stackrel{!}{=}0 \\ \frac{\partial\mathcal{L}}{\partial\phi_2^1(d)} &= \mathbf{P}(du)\cdot v'(\phi_2^0(d)+\phi_2^1(d)\cdot S_2^1(du))\cdot S_2^1(du) \\ &+\mathbf{P}(dd)\cdot v'(\phi_2^0(d)+\phi_2^1(d)\cdot S_2^1(dd))\cdot S_2^1(dd) \\ &-\textstyle\sum_{j=1}^J\lambda_j\cdot\left[\mathbf{Q}_j^B(du)\cdot S_2^1(du)+\mathbf{Q}_j^B(dd)\cdot S_2^1(dd)\right]\stackrel{!}{=}0\end{aligned}\right. \tag{6.42}$$

$$\left\{\begin{aligned}\frac{\partial\mathcal{L}}{\partial\phi_2^0(u)} &= \mathbf{P}(uu)\cdot v'(\phi_2^0(u)+\phi_2^1(u)\cdot S_2^1(uu)) \\ &+\mathbf{P}(ud)\cdot v'(\phi_2^0(u)+\phi_2^1(u)\cdot S_2^1(ud)) \\ &-\textstyle\sum_{j=1}^J\lambda_j\cdot\left[\mathbf{Q}_j^B(uu)+\mathbf{Q}_j^B(ud)\right]\stackrel{!}{=}0 \\ \frac{\partial\mathcal{L}}{\partial\phi_2^0(d)} &= \mathbf{P}(du)\cdot v'(\phi_2^0(d)+\phi_2^1(d)\cdot S_2^1(du)) \\ &+\mathbf{P}(dd)\cdot v'(\phi_2^0(d)+\phi_2^1(d)\cdot S_2^1(ud)) \\ &-\textstyle\sum_{j=1}^J\lambda_j\cdot\left[\mathbf{Q}_j^B(du)+\mathbf{Q}_j^B(dd)\right]\stackrel{!}{=}0\end{aligned}\right. \tag{6.43}$$

$$\left\{\ \forall j : \frac{\partial\mathcal{L}}{\partial\lambda_j} = W_0^\pi-\mathbf{E}_0^{\mathbf{Q}_j}[\phi_2^0+\phi_2^1\cdot S_2^1]\stackrel{!}{=}0\right.$$

Adding up the first order conditions for the stock, as provided in (6.42), yields,

$$\mathbf{E}_0^{\mathbf{P}}\left[v'(\phi_2^0 + \phi_2^1 \cdot S_2^1) \cdot S_2^1\right] = \sum_{j=1}^{J} \lambda_j \cdot \mathbf{E}_0^{\mathbf{Q}_j^B}\left[S_2^1\right] = \sum_{j=1}^{J} \lambda_j \cdot S_0^1. \qquad (6.44)$$

Doing the same with the first order conditions with respect to the bond - this time found in (6.43) - yields,

$$\mathbf{E}_0^{\mathbf{P}}\left[v'(\phi_2^0 + \phi_2^1 \cdot S_2^1)\right] = \sum_{j=1}^{J} \lambda_j. \qquad (6.45)$$

Dividing (6.44) by (6.45), one eventually arrives at an explicit expression for the stock price S_0^1,

$$\begin{aligned} S_0^1 &= \frac{\mathbf{E}_0^{\mathbf{P}}\left[v'(\phi_2^0 + \phi_2^1 \cdot S_2^1) \cdot S_2^1\right]}{\mathbf{E}_0^{\mathbf{P}}\left[v'(\phi_2^0 + \phi_2^1 \cdot S_2^1)\right]} \\ \Rightarrow S_0 &= \frac{\mathbf{E}_0^{\mathbf{P}}\left[v'(W_2) \cdot S_2^1\right]}{\mathbf{E}_0^{\mathbf{P}}\left[v'(W_2)\right]}. \end{aligned}$$

Step 4: The last little step is to realize that in general equilibrium the aggregate endowment η_2 at $n = 2$ has to be divided between the two groups of agents,

$$\eta_2 = W_2 + A_2, \qquad (6.46)$$

where W_2 denotes the non-hedgers' aggregate payoff at $n = 2$ and A_2 denotes the hedgers' aggregate payoff. Due to strict monotonicity of the utility function $v(\cdot)$ it is clear that the non-hedgers consume all their available $n = 2$ wealth. In view of (6.46), the $n = 2$ wealth of the non-hedgers is,

$$W_2 = \eta_2 - A_2,$$

where $\eta_2 - A_2 > 0$ by assumption. It is worth pointing out that A_2 is unique and independent of the equilibrium prices at dates $n = 0$ and $n = 1$ as shown in sub-section 6.2.3. Hence, the unique $n = 0$ equilibrium stock price must satisfy,

$$S_0^1 = \frac{\mathbf{E}_0^{\mathbf{P}}\left[v'(\eta_2 - A_2) \cdot \eta_2\right]}{\mathbf{E}_0^{\mathbf{P}}\left[v'(\eta_2 - A_2)\right]},$$

where η_2 replaces S_2^1. This completes the proof for $n = 0$.

Step 3': The proof for $n = 1$ is easier. Consider, for instance, the problem of the non-hedgers at the u node,

$$\max_{\phi_2^0(u),\phi_2^1(u)} \mathbf{E}_{1u}^{\mathbf{P}}[v(\phi_2^0(u) + \phi_2^1(u) \cdot S_2^1)]$$
$$\text{s.t. } \phi_2^0(u) + \phi_2^1(u) \cdot S_1^1(u) = \phi_1^0 + \phi_1^1 \cdot S_1^1(u).$$

$\mathbf{E}_{1u}^{\mathbf{P}}$ denotes conditional expectation under $\mathbf{P}$ given the information set $\mathcal{F}_1$ at the u node at $n = 1$. Fortunately, expected utility maximizing agents decide dynamically consistent. Since the non-hedgers have this convenient characteristic, we can be assured that this problem delivers the same optimal values, $\phi_2^0(u)$ and $\phi_2^1(u)$, as those already determined by (6.40) and (6.41). And so we solve the agents' problem at the u node knowing that it yields the same optimal values for $\phi_2^0(u)$ and $\phi_2^0(d)$ as the problem at $n = 0$. Again we determine the first order conditions from the corresponding LAGRANGIAN function,

$$\begin{cases} \frac{\partial\mathcal{L}}{\partial\phi_2^1(u)} = \frac{1}{\mathbf{P}(uu)+\mathbf{P}(ud)} \cdot \big[\mathbf{P}(uu) \cdot v'(\phi_2^0(u) + \phi_2^1(u) \cdot S_2^1(uu)) \cdot S_2^1(uu) \\ \quad +\mathbf{P}(ud) \cdot v'(\phi_2^0(u) + \phi_2^1(u) \cdot S_2^1(ud)) \cdot S_2^1(ud)\big] - \lambda \cdot S_1^1(u) \stackrel{!}{=} 0 \end{cases}$$

$$\begin{cases} \frac{\partial\mathcal{L}}{\partial\phi_2^1(u)} = \frac{1}{\mathbf{P}(uu)+\mathbf{P}(ud)} \cdot \big[\mathbf{P}(uu) \cdot v'(\phi_2^0(u) + \phi_2^1(u) \cdot S_2^1(uu)) \\ \quad +\mathbf{P}(ud) \cdot v'(\phi_2^0(u) + \phi_2^1(u) \cdot S_2^1(ud))\big] - \lambda \stackrel{!}{=} 0 \end{cases}$$

$$\begin{cases} \frac{\partial\mathcal{L}}{\partial\lambda} = \phi_2^0(u) + \phi_2^1(u) \cdot S_1^1(u) - \phi_1^0 - \phi_1^1 \cdot S_1^1(u) \stackrel{!}{=} 0 \end{cases}$$

Dividing the first condition by the second eventually yields,

$$S_1^1(u) = \frac{\mathbf{E}_{1u}^{\mathbf{P}}\left[v'\left(W_2\right) \cdot \eta_2\right]}{\mathbf{E}_{1u}^{\mathbf{P}}\left[v'\left(W_2\right)\right]}.$$

Step 4': After substituting for $W_2 = \eta_2 - A_2$,

$$S_1^1(u) = \frac{\mathbf{E}_{1u}^{\mathbf{P}}\left[v'\left(\eta_2 - A_2\right) \cdot \eta_2\right]}{\mathbf{E}_{1u}^{\mathbf{P}}\left[v'\left(\eta_2 - A_2\right)\right]},$$

as desired.

The calculations to prove the assertion for the d node are very similar so we can omit them. This completes the proof. ■

6.7.2 Proof of proposition 50

The proof consists of two steps. First, we verify that $\mathbf{Q}^*$ is a $\mathbf{P}-$equivalent probability measure. Second, we prove that $\mathbf{Q}^*$ is a martingale measure.

Step 1: Note that (6.19) indeed defines a $\mathbf{P}-$equivalent probability measure, i.e., it satisfies,

$$\forall\omega \in \Omega : \mathbf{Q}^*(\omega) > 0 \text{ and}$$
$$\sum_{\omega\in\Omega} \mathbf{Q}^*(\omega) = 1.$$

The second property is obtained by straightforward manipulations,

$$\begin{aligned}\sum_{\omega\in\Omega} \mathbf{Q}^*(\omega) &= \sum_{\omega\in\Omega} \frac{\mathbf{P}(\omega)\cdot v'(\eta_2^\omega - A_2(\omega))}{\mathbf{E}_0^{\mathbf{P}}\left[v'(\eta_2 - A_2)\right]} \\ &= \mathbf{E}_0^{\mathbf{P}}\left[\frac{v'(\eta_2 - A_2)}{\mathbf{E}_0^{\mathbf{P}}\left[v'(\eta_2 - A_2)\right]}\right] \\ &= \frac{\mathbf{E}_0^{\mathbf{P}}\left[v'(\eta_2 - A_2)\right]}{\mathbf{E}_0^{\mathbf{P}}\left[v'(\eta_2 - A_2)\right]} = 1,\end{aligned}$$

where η_2^ω and $A_2(\omega)$ denote aggregate supply of the homogenous good and aggregate hedge demand in state $\omega \in \Omega$ at $n = 2$, respectively.

Step 2: It remains to verify that $\mathbf{Q}^*$ as defined in (6.19) is indeed a martingale measure. To do so, we have to show that $\mathbf{E}_0^{\mathbf{Q}^*}[\eta_2] = S_0^1$ as well as $\mathbf{E}_1^{\mathbf{Q}^*}[\eta_2] = S_1^1$. And fortunately,

$$\begin{aligned}\mathbf{E}_0^{\mathbf{Q}^*}[\eta_2] &= \sum_{\omega\in\Omega} \mathbf{Q}^*(\omega)\cdot\eta_2^\omega \\ &= \sum_{\omega\in\Omega} \mathbf{P}(\omega)\cdot\frac{v'(\eta_2^\omega - A_2(\omega))}{\mathbf{E}_0^{\mathbf{P}}\left[v'(\eta_2 - A_2)\right]}\cdot\eta_2^\omega \\ &= \mathbf{E}_0^{\mathbf{P}}\left[\frac{v'(\eta_2 - A_2)}{\mathbf{E}_0^{\mathbf{P}}\left[v'(\eta_2 - A_2)\right]}\cdot\eta_2\right] \\ &= \frac{\mathbf{E}_0^{\mathbf{P}}\left[v'(\eta_2 - A_2)\cdot\eta_2\right]}{\mathbf{E}_0^{\mathbf{P}}\left[v'(\eta_2 - A_2)\right]} \\ &= S_0^1,\end{aligned}$$

as desired. The last equality follows from Theorem 49. We demonstrate the martingale property for the u node at $n = 1$ only.

$$\begin{aligned}&\mathbf{E}_{1u}^{\mathbf{Q}^*}[\eta_2] \\ &= \sum_{\omega\in\{uu,ud\}} \frac{\mathbf{Q}^*(\omega)}{\mathbf{Q}^*(uu)+\mathbf{Q}^*(ud)}\cdot\eta_2^\omega \\ &= \sum_{\omega\in\{uu,ud\}} \left[\frac{\mathbf{P}(\omega)\cdot v'(\eta_2^\omega - A_2(\omega)}{\mathbf{E}_0^{\mathbf{P}}\left[v'(\eta_2 - A_2)\right]}\right.\end{aligned}$$

$$
\begin{aligned}
&\cdot\left(\frac{\mathbf{P}(uu)\cdot v'(\eta_2^{uu}-A_2(uu))+\mathbf{P}(ud)\cdot v'(\eta_2^{ud}-A_2(ud)}{\mathbf{E}_0^{\mathbf{P}}\left[v'(\eta_2-A_2)\right]}\right)^{-1}\cdot\eta_2^{\omega}\Bigg]\\
&=\sum_{\omega\in\{uu,ud\}}\frac{\mathbf{P}(\omega)\cdot v'(\eta_2^{\omega}-A_2^{\omega})}{\mathbf{P}(uu)\cdot v'(\eta_2^{uu}-A_2(uu))+\mathbf{P}(ud)\cdot v'(\eta_2^{ud}-A_2(ud))}\cdot\eta_2^{\omega}\\
&=\frac{\mathbf{P}(uu)\cdot v'(\eta_2^{uu}-A_2(uu))\cdot\eta_2^{uu}+\mathbf{P}(ud)\cdot v'(\eta_2^{ud}-A_2(ud))\cdot\eta_2^{ud}}{\mathbf{P}(uu)\cdot v'(\eta_2^{uu}-A_2(uu))+\mathbf{P}(ud)\cdot v'(\eta_2^{ud}-A_2(ud))}\\
&\qquad\cdot\frac{\mathbf{P}(uu)+\mathbf{P}(ud)}{\mathbf{P}(uu)+\mathbf{P}(ud)}\\
&=\frac{\mathbf{E}_{1u}^{\mathbf{P}}[v'(\eta_2-A_2)\cdot\eta_2]}{\mathbf{E}_{1u}^{\mathbf{P}}[v'(\eta_2-A_2)]}\\
&=S_1^1(u).
\end{aligned}
$$

$\mathbf{E}_{1u}^{\mathbf{P}}$ denotes conditional expectation given the information set $\mathcal{F}_1$ at the u node at $n=1$. The last equality again follows from Theorem 49. ■

6.7.3 Proof of proposition 55

Using the general pricing equation (6.18) and recalling that $\eta_2^{uu}>\overline{K}>\eta_2^{ud}$, the $n=0$ equilibrium price of the stock is,

$$
S_0^1=\frac{\mathbf{E}_0^{\mathbf{P}}\left[v'(\eta_2-\alpha\cdot\overline{C}_2)\cdot\eta_2\right]}{\mathbf{E}_0^{\mathbf{P}}\left[v'(\eta_2-\alpha\cdot\overline{C}_2)\right]},\tag{6.47}
$$

where,

$$
\begin{aligned}
\mathbf{E}_0^{\mathbf{P}}\left[v'(\eta_2-\alpha\cdot\overline{C}_2)\cdot\eta_2\right] &= \mathbf{P}(uu)\cdot v'(\eta_2^{uu}-\alpha\cdot\overline{C}_2(uu))\cdot\eta_2^{uu}\\
&\quad+\mathbf{P}(ud)\cdot v'(\eta_2^{ud})\cdot\eta_2^{ud}\\
&\quad+\mathbf{P}(du)\cdot v'(\eta_2^{du})\cdot\eta_2^{du}\\
&\quad+\mathbf{P}(dd)\cdot v'(\eta_2^{dd})\cdot\eta_2^{dd},
\end{aligned}
$$

and,

$$
\begin{aligned}
\mathbf{E}_0^{\mathbf{P}}\left[v'(\eta_2-\alpha\cdot\overline{C}_2)\cdot\eta_2\right] &= \mathbf{P}(uu)\cdot v'(\eta_2^{uu}-\alpha\cdot\overline{C}_2(uu))\\
&\quad+\mathbf{P}(ud)\cdot v'(\eta_2^{ud})\\
&\quad+\mathbf{P}(du)\cdot v'(\eta_2^{du})\\
&\quad+\mathbf{P}(dd)\cdot v'(\eta_2^{dd}).
\end{aligned}
$$

Differentiating (6.47) with respect to α delivers,

$$\begin{aligned}
\frac{\partial S_0^1}{\partial \alpha} &= \frac{\partial}{\partial \alpha}\frac{\mathbf{E}_0^{\mathbf{P}}\left[v'(\eta_2-\alpha\cdot\overline{C}_2)\cdot\eta_2\right]}{\mathbf{E}_0^{\mathbf{P}}\left[v'(\eta_2-\alpha\cdot\overline{C}_2)\right]} \\
&= \frac{\left[\mathbf{P}(uu)\cdot v''(\eta_2^{uu}-\alpha\cdot\overline{C}_2(uu))\cdot\eta_2^{uu}\cdot(-\overline{C}_2(uu))\right]\cdot\mathbf{E}_0^{\mathbf{P}}\left[v'(\eta_2-\alpha\cdot\overline{C}_2)\right]}{\mathbf{E}_0^{\mathbf{P}}\left[v'(\eta_2-\alpha\cdot\overline{C}_2)\right]^2} \\
&\quad -\frac{\mathbf{E}_0^{\mathbf{P}}\left[v'(\eta_2-\alpha\cdot\overline{C}_2)\cdot\eta_2\right]\cdot\left[\mathbf{P}(uu)\cdot v''(\eta_2^{uu}-\alpha\cdot\overline{C}_2(uu))\cdot(-\overline{C}_2(uu))\right]}{\mathbf{E}_0^{\mathbf{P}}\left[v'(\eta_2-\alpha\cdot\overline{C}_2)\right]^2} \\
&= \frac{\overbrace{\mathbf{P}(uu)\cdot\mathbf{P}(ud)\cdot v'(\eta_2^{ud})\cdot v''(\eta_2^{uu}-\alpha\cdot\overline{C}_2(uu))\cdot(-\overline{C}_2(uu))}^{>0}\cdot\overbrace{\left[\eta_2^{uu}-\eta_2^{ud}\right]}^{>0}}{\mathbf{E}_0^{\mathbf{P}}\left[v'(\eta_2-\alpha\cdot\overline{C}_2)\right]^2} \\
&\quad +\frac{\overbrace{\mathbf{P}(uu)\cdot\mathbf{P}(du)\cdot v'(\eta_2^{du})\cdot v''(\eta_2^{uu}-\alpha\cdot\overline{C}_2(uu))\cdot(-\overline{C}_2(uu))}^{>0}\cdot\overbrace{\left[\eta_2^{uu}-\eta_2^{du}\right]}^{>0}}{\mathbf{E}_0^{\mathbf{P}}\left[v'(\eta_2-\alpha\cdot\overline{C}_2)\right]^2} \\
&\quad +\frac{\overbrace{\mathbf{P}(uu)\cdot\mathbf{P}(dd)\cdot v'(\eta_2^{dd})\cdot v''(\eta_2^{uu}-\alpha\cdot\overline{C}_2(uu))\cdot(-\overline{C}_2(uu))}^{>0}\cdot\overbrace{\left[\eta_2^{uu}-\eta_2^{dd}\right]}^{>0}}{\mathbf{E}_0^{\mathbf{P}}\left[v'(\eta_2-\alpha\cdot\overline{C}_2)\right]^2} \\
&> 0.
\end{aligned}$$

Consequently, $\frac{\partial S_0^1}{\partial \alpha} > 0$ implying that the $n = 0$ equilibrium price S_0^1 of the stock increases with increasing market weight α of the hedgers. This proves the proposition. ■

6.7.4 Proof of proposition 57

According to the general pricing equation (6.18) and $\eta_2^{uu} > \overline{K} > \eta_2^{ud}$, the $n = 1$ equilibrium stock price $S_1^1(u)$ at the u node is,

$$\begin{aligned}
S_1^1(u) &= \frac{\mathbf{E}_{1u}^{\mathbf{P}}\left[v'(\eta_2-\alpha\cdot\overline{C}_2)\cdot\eta_2\right]}{\mathbf{E}_{1u}^{\mathbf{P}}\left[v'(\eta_2-\alpha\cdot\overline{C}_2)\right]} \\
&= \frac{\frac{1}{\mathbf{P}(uu)+\mathbf{P}(ud)}\cdot\left[\mathbf{P}(uu)\cdot v'\left(\eta_2^{uu}-\alpha\cdot\overline{C}_2(uu)\right)\cdot\eta_2^{uu}+\mathbf{P}(ud)\cdot v'\left(\eta_2^{ud}\right)\cdot\eta_2^{ud}\right]}{\frac{1}{\mathbf{P}(uu)+\mathbf{P}(ud)}\cdot\left[\mathbf{P}(uu)\cdot v'\left(\eta_2^{uu}-\alpha\cdot\overline{C}_2(uu)\right)+\mathbf{P}(ud)\cdot v'\left(\eta_2^{ud}\right)\right]},
\end{aligned}$$

because the only strictly positive payoff of the call option occurs in the best state. Differentiating this expression with respect to α, one gets,

$$\frac{\partial S_1^1(u)}{\partial \alpha}$$

$$= \frac{\left[\frac{\mathbf{P}(uu)}{\mathbf{P}(uu)+\mathbf{P}(ud)} \cdot v''(\eta_2^{uu} - \alpha \cdot \overline{C}_2(uu)) \cdot (-\overline{C}_2(uu)) \cdot \eta_2^{uu}\right] \cdot \mathbf{E}_{1u}^{\mathbf{P}}\left[v'(\eta_2 - \alpha \cdot \overline{C}_2)\right]}{\mathbf{E}_{1u}^{\mathbf{P}}\left[v'(\eta_2 - \alpha \cdot \overline{C}_2)\right]^2}$$
$$- \frac{\mathbf{E}_{1u}^{\mathbf{P}}\left[v'(\eta_2 - \alpha \cdot \overline{C}_2) \cdot \eta_2\right] \cdot \left[\frac{\mathbf{P}(uu)}{\mathbf{P}(uu)+\mathbf{P}(ud)} \cdot v''(\eta_2^{uu} - \alpha \cdot \overline{C}_2(uu)) \cdot (-\overline{C}_2(uu))\right]}{\mathbf{E}_{1u}^{\mathbf{P}}\left[v'(\eta_2 - \alpha \cdot \overline{C}_2)\right]^2}$$
$$= \mathbf{E}_{1u}^{\mathbf{P}}\left[v'(\eta_2 - \alpha \cdot \overline{C}_2) \cdot \eta_2\right]^{-2}$$
$$\cdot \underbrace{\left[\frac{\mathbf{P}(uu)}{\mathbf{P}(uu)+\mathbf{P}(ud)} \cdot v''(\eta_2^{uu} - \alpha \cdot \overline{C}_2(uu)) \cdot (-\overline{C}_2(uu))\right]}_{>0}$$
$$\cdot \left(\eta_2^{uu} \cdot \mathbf{E}_{1u}^{\mathbf{P}}\left[v'(\eta_2 - \alpha \cdot \overline{C}_2)\right] - \mathbf{E}_{1u}^{\mathbf{P}}\left[v'(\eta_2 - \alpha \cdot \overline{C}_2) \cdot \eta_2\right]\right) \tag{6.48}$$

To decide upon the sign of (6.48), it is necessary to check the sign of the last term in parentheses,

$$\eta_2^{uu} \cdot \mathbf{E}_{1u}^{\mathbf{P}}\left[v'(\eta_2 - \alpha \cdot \overline{C}_2)\right] - \mathbf{E}_{1u}^{\mathbf{P}}\left[v'(\eta_2 - \alpha \cdot \overline{C}_2) \cdot \eta_2\right]$$
$$= \frac{1}{\mathbf{P}(uu)+\mathbf{P}(ud)}$$
$$\cdot \left[\mathbf{P}(uu) \cdot v'\left(\eta_2^{uu} - \alpha \cdot \overline{C}_2(uu)\right) \cdot \eta_2^{uu} + \mathbf{P}(ud) \cdot v'\left(\eta_2^{ud}\right) \cdot \eta_2^{uu}\right.$$
$$\left. - \mathbf{P}(uu) \cdot v'\left(\eta_2^{uu} - \alpha \cdot \overline{C}_2(uu)\right) \cdot \eta_2^{uu} - \mathbf{P}(ud) \cdot v'\left(\eta_2^{ud}\right) \cdot \eta_2^{ud}\right]$$
$$= \frac{1}{\mathbf{P}(uu)+\mathbf{P}(ud)} \cdot \mathbf{P}(ud) \cdot v'\left(\eta_2^{ud}\right) \cdot \underbrace{\left[\eta_2^{uu} - \eta_2^{ud}\right]}_{>0} > 0. \tag{6.49}$$

Finally, (6.49) in combination with (6.48) implies $\frac{\partial S_1^1(u)}{\partial \alpha} > 0$. The stock price $S_1^1(u)$ increases with increasing market weight α of hedgers dynamically hedging calls $\overline{C}_2$. ■

6.7.5 Proof of proposition 61

From the proof of proposition 57 and (6.37), one can conclude that in case (b),

$$\frac{\partial S_1^1(u)}{\partial \alpha} \geq 0,$$

holds. Equality is observed if $\rho = 0$. This proves the first part of the assertion. However, we additionally have to determine the effect on the stock price $S_1^1(d)$. It is clear that for $\rho = 1$, $\frac{\partial S_1^1(d)}{\partial \alpha} = 0$. Therefore, assume for the moment that $\rho < 1$. According to the pricing equation (6.18) and the assumption $\eta_2^{du} > \overline{X} > \eta_2^{dd}$, the $n = 1$ equilibrium stock price $S_1^1(d)$ at the d

node is,

$$
\begin{aligned}
S_1^1(d) &= \frac{\mathbf{E}_{1d}^{\mathbf{P}}\left[v'(\eta_2 - \alpha \cdot \overline{A}_2) \cdot \eta_2\right]}{\mathbf{E}_{1d}^{\mathbf{P}}\left[v'(\eta_2 - \alpha \cdot \overline{A}_2)\right]} \\
&= \frac{\frac{1}{\mathbf{P}(du)+\mathbf{P}(dd)} \cdot \left[\mathbf{P}(du) \cdot v'\left(\eta_2^{du}\right) \cdot \eta_2^{du} + \mathbf{P}(dd) \cdot v'\left(\eta_2^{dd} - \alpha \cdot \overline{A}_2(dd)\right) \cdot \eta_2^{dd}\right]}{\frac{1}{\mathbf{P}(du)+\mathbf{P}(dd)} \cdot \left[\mathbf{P}(du) \cdot v'\left(\eta_2^{du}\right) + \mathbf{P}(dd) \cdot v'\left(\eta_2^{dd} - \alpha \cdot \overline{A}_2(dd)\right)\right]},
\end{aligned}
$$

because a strictly positive payoff occurs in the dd state only [see (6.37)]. $\mathbf{E}_{1d}^{\mathbf{P}}$ denotes conditional expectation given the information set $\mathcal{F}_1$ at the d node at $n = 1$. Differentiating this expression with respect to α yields,

$$
\begin{aligned}
&\frac{\partial S_1^1(d)}{\partial \alpha} \\
&= \frac{\left[\frac{\mathbf{P}(dd)}{\mathbf{P}(du)+\mathbf{P}(dd)} \cdot v''(\eta_2^{dd} - \alpha \cdot \overline{A}_2(dd)) \cdot (-\overline{A}_2(dd)) \cdot \eta_2^{dd}\right] \cdot \mathbf{E}_{1d}^{\mathbf{P}}\left[v'(\eta_2 - \alpha \cdot \overline{A}_2)\right]}{\mathbf{E}_{1d}^{\mathbf{P}}\left[v'(\eta_2 - \alpha \cdot \overline{A}_2)\right]^2} \\
&\quad - \frac{\mathbf{E}_{1u}^{\mathbf{P}}\left[v'(\eta_2 - \alpha \cdot \overline{A}_2) \cdot \eta_2\right] \cdot \left[\frac{\mathbf{P}(dd)}{\mathbf{P}(du)+\mathbf{P}(dd)} \cdot v''(\eta_2^{dd} - \alpha \cdot \overline{A}_2(dd)) \cdot (-\overline{A}_2(dd))\right]}{\mathbf{E}_{1d}^{\mathbf{P}}\left[v'(\eta_2 - \alpha \cdot \overline{A}_2)\right]^2} \\
&= \mathbf{E}_{1d}^{\mathbf{P}}\left[v'(\eta_2 - \alpha \cdot \overline{A}_2)\right]^{-2} \\
&\quad \cdot \underbrace{\left[\frac{\mathbf{P}(dd)}{\mathbf{P}(du) + \mathbf{P}(dd)} \cdot v''(\eta_2^{dd} - \alpha \cdot \overline{A}_2(dd)) \cdot (-\overline{A}_2(dd))\right]}_{>0} \\
&\quad \cdot \left(\eta_2^{dd} \cdot \mathbf{E}_{1d}^{\mathbf{P}}\left[v'(\eta_2 - \alpha \cdot \overline{A}_2)\right] - \mathbf{E}_{1d}^{\mathbf{P}}\left[v'(\eta_2 - \alpha \cdot \overline{A}_2) \cdot \eta_2\right]\right)
\end{aligned}
$$

The term in the last line has negative sign, as one can see from,

$$
\begin{aligned}
&\eta_2^{dd} \cdot \mathbf{E}_{1d}^{\mathbf{P}}\left[v'(\eta_2 - \alpha \cdot \overline{A}_2)\right] - \mathbf{E}_{1d}^{\mathbf{P}}\left[v'(\eta_2 - \alpha \cdot \overline{A}_2) \cdot \eta_2\right] \\
&= \frac{1}{\mathbf{P}(du) + \mathbf{P}(dd)} \cdot \left[\mathbf{P}(du) \cdot v'\left(\eta_2^{du}\right) \cdot \eta_2^{dd} + \mathbf{P}(dd) \cdot v'\left(\eta_2^{dd} - \alpha \cdot \overline{A}_2(dd)\right) \cdot \eta_2^{dd}\right. \\
&\quad \left. - \mathbf{P}(du) \cdot v'\left(\eta_2^{du}\right) \cdot \eta_2^{du} - \mathbf{P}(dd) \cdot v'\left(\eta_2^{dd} - \alpha \cdot \overline{A}_2(dd)\right) \cdot \eta_2^{dd}\right] \\
&= \frac{\mathbf{P}(du)}{\mathbf{P}(du) + \mathbf{P}(dd)} \cdot v'\left(\eta_2^{du}\right) \cdot \underbrace{\left[\eta_2^{dd} - \eta_2^{du}\right]}_{<0} < 0.
\end{aligned}
$$

Everything taken together, this implies $\frac{\partial S_1^1(d)}{\partial \alpha} \leq 0$ for $\rho \in [0, 1]$. In summary, the stock price $S_1^1(u)$ is *non-decreasing* in α, whereas the stock price $S_1^1(d)$ is *non-increasing* in α. For $\rho > 0$ the stock price $S_1^1(u)$ is *strictly increasing* in α, and for $\rho < 1$ the stock price $S_1^1(d)$ is *strictly decreasing* in α. ■

Chapter 7

Dynamic hedging and general equilibrium in incomplete markets

7.1 Introduction

As seen in chapter 2, the impact of dynamic hedging on financial market equilibrium has become a growing area of research in financial economics. One feature that such studies typically have in common is that markets are complete a priori and remain so under a common knowledge assumption regarding the hedging activity in the marketplace. As a crucial implication, dynamic hedging strategies are uniquely determined. In other words, hedgers act as automata in complete markets, only executing a predetermined hedge program. This was also the case in the previous chapter. Since every contingent claim was attainable in the market model $\mathcal{M}^{cm}$, hedgers could always achieve a perfect hedge at given costs.

In incomplete markets, dynamic hedging strategies are not uniquely determined like in complete markets. Because perfect hedges are no longer feasible, a hedger needs to impose some kind of objective function in order to decide upon the *optimal* hedge strategy. Several authors have developed different approaches to the dynamic hedging of derivatives in incomplete markets. One example is the quadratic hedging approach where the expected quadratic deviation of the hedge portfolio's payoff from the derivative's payoff at maturity is minimized.[1] Another one is the approach which seeks to minimize the expected loss from hedging.[2] When introducing the hedgers

[1] For example, FREY (1997) provides a brief introduction to this approach.
[2] See, for instance, CVITANIC (1998).

in the last chapter, we required that they implement the admissible trading strategy that minimizes the costs of super-replicating a given contingent claim. Yet in the last chapter this requirement boiled down to perfect replication because the market model in chapter 6 was complete in equilibrium. In this chapter, super-replication will play a central role because perfect hedges are infeasible in general. Recently, the super-replication approach to hedging contingent claims in incomplete markets has become very popular in the theoretical literature. It is, for example, extensively discussed in chapter 5 of Karatzas and Shreve (1998).[3]

The analysis of dynamic hedging strategies in a general equilibrium framework where markets are incomplete by assumption has not received any attention so far. This is somewhat surprising since there is no doubt that incomplete markets draw a more realistic picture of the world than complete markets. Furthermore, the theory of incomplete markets presently represents one of the most active fields of research in financial economics.[4] Trying to close the apparently existing gap, we investigate in this chapter dynamic hedging in a general equilibrium framework with incomplete markets. Technically, incomplete markets imply that there exists a non-empty subset of the set of all contingent claims, elements of which are not attainable via a self-financing trading strategy. We embed the analysis in a discrete time, discrete space general equilibrium framework, which is essentially a variation of the market model $\mathcal{M}^{cm}$ as proposed in chapter 6.

The chapter is structured as follows. We explore the market model in section 7.2. Section 7.3 derives optimal super-replication strategies for European call and put options. The analysis of general equilibrium takes place in section 7.4. Section 7.5 carries out the comparative statics analysis while numerical computations are conducted in section 7.6. Section 7.7 summarizes the main results and section 7.8 provides proof of some results found in the other sections.

7.2 The market model

The market model in which we embed the analysis in this chapter is a modification of the market model $\mathcal{M}^{cm}$ of the last chapter. In a sense, it is a generalization of that market model since it is characterized by market incompleteness. This section outlines the main differences between the market model $\mathcal{M}^{cm}$ and the market model which the analysis in this chapter is based on.

[3] Refer also to Cvitanic (1997), Cvitanic, Pham, and Touzi (1997) or Frey (1999).

[4] Magill and Quinzii (1996) give an overview of recent advances.

The following assumptions remain in force: *perfect markets*, *perfect competition, complete* and *symmetric information*, *one homogenous consumption good.*

7.2.1 Primitives

Consider the market model $\mathcal{M}^{cm} = \{(\Omega, \wp(\Omega), \mathbb{F}, \mathbf{P}), N = 2, \mathbb{S}^1, \mathbb{I}^\alpha\}$ where uncertainty resolves according to the fundamental state process $(\eta_n)_{n\in\{0,1,2\}}$, $\forall n : \eta_n > 0$, that has now an event tree representation of,

first date	intermediate date	terminal date
		η_2^{uu}
	η_1^u	η_2^{um}
		η_2^{ud}
η_0		
		η_2^{du}
	η_1^d	η_2^{dm}
		η_2^{dd}
$n = 0$	$n = 1$	$n = 2$

We assume,

$$\eta_2^{uu} > \eta_2^{um} > \eta_2^{ud} \geq \eta_2^{du} > \eta_2^{dm} > \eta_2^{dd}.$$

Correspondingly, the state space enlarges to $\Omega = \{uu, um, ud, du, dm, dd\}$. Even though six states ω of the economy are now possible at the terminal date, there are still only two states possible at the intermediate date. The terminal nodes $\{uu, um, ud\}$ may only be reached from the u node at $n = 1$ while the nodes $\{du, dm, dd\}$ may only be reached from the d node at $n = 1$. This ensures that we can directly compare the comparative statics results regarding stock price volatility of this chapter with those obtained in the previous chapter. The filtration according to which agents learn information is $\mathbb{F} = (\mathcal{F}_n)_{n\in\{0,1,2\}}$, where now,

$$\begin{aligned}
\mathcal{F}_0 &= \{\varnothing, \Omega\}, \\
\mathcal{F}_1 &= \{\varnothing, \{uu, um, ud\}, \{du, dm, dd\}, \Omega\} \text{ and} \\
\mathcal{F}_2 &= \wp(\Omega).
\end{aligned}$$

As in the market model $\mathcal{M}^{cm}$, the probability measure $\mathbf{P}$ is strictly positive for all $\omega \in \Omega$, i.e., $\forall\omega \in \Omega : \mathbf{P}(\omega) > 0$. Altogether, this defines the new filtered probability space $(\Omega, \wp(\Omega), \mathbb{F}, \mathbf{P})$ which with we will work.

7.2.2 Securities

The stock price process $(S_n^1)_{n\in\{0,1,2\}}$ is adapted to the new filtration $\mathbb{F}$. Therefore, it evolves as displayed below.

first date	intermediate date	terminal date
		$S_2^1(uu) = \eta_2^{uu}$
	$S_1^1(u)$	$S_2^1(um) = \eta_2^{um}$
		$S_2^1(ud) = \eta_2^{ud}$
S_0^1		
		$S_2^1(du) = \eta_2^{du}$
	$S_1^1(d)$	$S_2^1(dm) = \eta_2^{dm}$
		$S_2^1(dd) = \eta_2^{dd}$
$n = 0$	$n = 1$	$n = 2$

We assume that the market model is *free of arbitrage opportunities* which ensures by using Theorem 29 that the set $\mathbb{Q}$ of $\mathbf{P}-$equivalent martingale measures will be non-empty in equilibrium.

7.2.3 Agents

There is still a continuum $\mathbb{I} = [0,1]$ of agents which divides into two groups, hedgers with market weight $\alpha \in [0,1[$ and non-hedgers with market weight $1-\alpha$. With respect to these groups of agents all assumptions made in chapter 6 remain in force. In particular, hedgers dynamically hedge contingent claims. They implement the admissible trading strategy that minimizes the costs of super-replicating a given contingent claim that they have sold OTC. This procedure was perfectly successful in the setting of chapter 6, in the sense that every contingent claim was attainable via an admissible trading strategy. This is due to the market model $\mathcal{M}^{cm}$ being complete under complete and symmetric information.

A somewhat different picture will emerge for the present market model. We will argue that it is incomplete by construction. As a result, there are contingent claims that are not attainable via admissible trading strategies so that perfect hedges are infeasible in general. However, if a *perfect* hedge is infeasible, it might be nonetheless possible that a *complete* hedge is feasible. By *complete* hedge we mean that there exists an admissible trading strategy that super-replicates the contingent claim.[5] Section 7.3 is devoted to issues

[5] We should note that we use the expression dynamic hedging for trading strategies that generate perfect hedges as well as for those that generate complete hedges. In that sense, perfect replication and super-replication are just special cases of dynamic hedging strategies.

related to super-replication.

Summary

The new market model is denoted,

$$\mathcal{M}^{im} = \{(\Omega, \wp(\Omega), \mathbb{F}, \mathbf{P}), N = 2, \mathbb{S}^1, \mathbb{I}^\alpha\},$$

where,

- $\Omega = \{uu, um, ud, du, dm, dd\}$,
- $\mathbb{F}$ is the filtration generated by the state process $(\eta_n)_{n\in\{0,1,2\}}$,
- $\mathbf{P}$ is strictly positive for all $\omega \in \Omega$,
- $N = 2$,
- $\mathbb{S}^1 = \{(S_n^k)_{n\in\{0,1,2\}} : k \in \{0,1\}\}$ where $\forall n : S_n^0 \equiv 1$ and
- $\mathbb{I}^\alpha = [0,1]$ with a proportion $\alpha \in [0,1[$ being hedgers and a proportion $1-\alpha$ being non-hedgers.

7.3 Super-replication strategies

In this section, we examine the hedgers' optimal super-replication strategies in detail. We first show in sub-section 7.3.1, that the optimal strategies are generally stock price-dependent and that there can be, in principle, an infinite number of solutions. The analysis proceeds then by deriving and characterizing the optimal super-replicating strategies for two particular contingent claims.

The first contingent claim, a European call option, is given by,

$$\overline{C}_2 : \mathbb{R}_{++} \to \mathbb{R}_+, S_2^1 \mapsto \max\{S_2^1 - K, 0\} \tag{7.1}$$
$$\text{where } \eta_2^{uu} > K \geq \eta_2^{um}. \tag{7.2}$$

The second contingent claim of interest is a European put option which is given by,

$$\overline{P}_2 : \mathbb{R}_{++} \to \mathbb{R}_+, S_2^1 \mapsto \max\{X - S_2^1, 0\} \tag{7.3}$$
$$\text{where } \eta_2^{dm} \geq X > \eta_2^{dd}. \tag{7.4}$$

The corresponding super-replication strategies are derived and characterized in sub-section 7.3.2. After having tackled equilibrium issues in section 7.4, section 7.5 conducts the comparative statics analysis in the market model $\mathcal{M}^{im}$ when hedgers hedge these particular options.

The notation used in the sequel is in accordance with the one of chapter 6.

7.3.1 Determinacy of optimal strategies

First note that the hedgers solve their super-replication problem through backward induction. To make our argument in this sub-section, it suffices to consider the sub-problem of the hedgers only at the u node at $n = 1$. For a general contingent claim $\overline{A}_2$, the super-replication sub-problem of the hedgers at the u node is,

$$\min_{\phi_2^0(u),\phi_2^1(u)} \phi_2^0(u) + \phi_2^1(u) \cdot S_1^1(u) \tag{7.5}$$

$$\text{s.t. } \phi_2^0(u) + \phi_2^1(u) \cdot S_2^1 \geq \widetilde{A}_2. \tag{7.6}$$

Here, we have defined $\widetilde{A}_2 \equiv \alpha \cdot \overline{A}_2$. (7.6) translates into,

$$\begin{aligned} \phi_2^0(u) &\geq \widetilde{A}_2(uu) - \phi_2^1(u) \cdot S_2^1(uu), & (7.7)\\ \phi_2^0(u) &\geq \widetilde{A}_2(um) - \phi_2^1(u) \cdot S_2^1(um), & (7.8)\\ \phi_2^0(u) &\geq \widetilde{A}_2(ud) - \phi_2^1(u) \cdot S_2^1(ud). & (7.9)\end{aligned}$$

For the sake of simplicity, we assume that $\widetilde{A}_2(uu) > \widetilde{A}_2(um) > \widetilde{A}_2(ud) = 0$. Any other relationship would be fine as well, but this one allows us to make our argument as simple as possible. In the following, we will analyze problem (7.5) and (7.6) graphically. We will conduct the analysis in a diagram where $\phi_2^0(u)$ is on the vertical axis and $\phi_2^1(u)$ is on the horizontal axis. For fixed super-replication costs $\widehat{A}_1$, the iso cost lines in such a diagram are determined by,

$$\phi_2^0(u) = \widehat{A}_1 - \phi_2^1(u) \cdot S_1^1(u).$$

Figure 7.1 illustrates problem (7.5) and (7.6). As it becomes obvious, two basic cases are possible. There can be one unique solution or infinitely many solutions. The latter case occurs if $S_1^1(u) = S_2^1(um)$. The former case if $S_1^1(u) \neq S_2^1(um)$. Depending on whether $S_1^1(u) > S_2^1(um)$ or $S_1^1(u) < S_2^1(um)$, the cost minimizing portfolio is determined by the intersection of,

$$\phi_2^0(u) = \widetilde{A}_2(uu) - \phi_2^1(u) \cdot S_2^1(uu),$$

and,

$$\phi_2^0(u) = \widetilde{A}_2(um) - \phi_2^1(u) \cdot S_2^1(um),$$

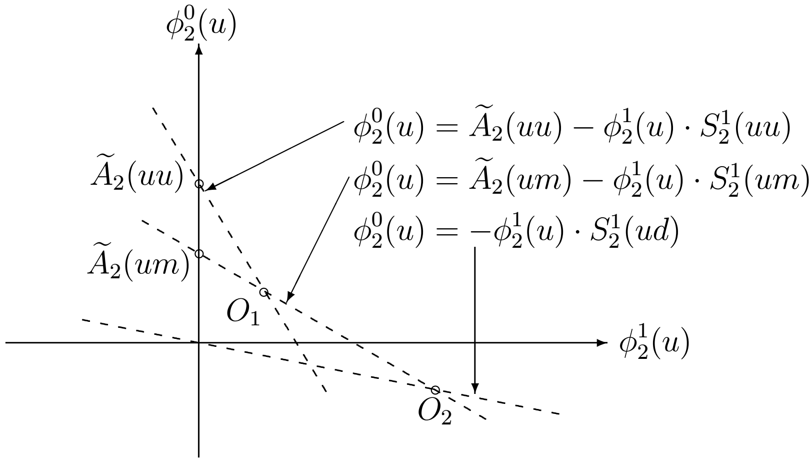

Figure 7.1: Derterminacy of the optimal solution.

in the former case and by,

$$\phi_2^0(u) = \widetilde{A}_2(um) - \phi_2^1(u) \cdot S_2^1(um),$$

and,

$$\phi_2^0(u) = -\phi_2^1(u) \cdot S_2^1(ud),$$

in the latter case. For $S_1^1(u) > S_2^1(um)$, the optimal portfolio is found in point O_1. Similarly, if $S_1^1(u) < S_2^1(um)$, the optimal portfolio is found in point O_2. If, however, $S_1^1(u) = S_2^1(um)$ holds, then all points on the line connecting O_1 and O_2 are optimal. The last case is, of course, not generic, but nonetheless possible and plausible.

If the contingent claim that the hedgers super-replicate has a different payoff structure, it might also be that the optimal strategy becomes independent of the stock price at date $n = 1$. Since this chapter, in the spirit of the last chapter, draws on differential calculus in the comparative statics analysis, it is tedious to consider multiple optimal solutions. Fortunately, the next sub-section will reveal that the optimal super-replication strategies for the call described by (7.1) and (7.2) and the put described by (7.3) and (7.4) are independent of the stock prices prior to $n = 2$. In the end, this particular feature will guarantee that there is a unique general equilibrium in the market model $\mathcal{M}^{im}$ when hedgers are obliged to hedge only these options.

7.3.2 Derivation of optimal strategies

In this sub-section, we will first demonstrate that neither the call option nor the put option are attainable in the market model $\mathcal{M}^{im}$. We will then derive the optimal super-replication strategies. The subsequent analysis presumes that there is a unique equilibrium stock price process, a result we will derive shortly.

A brief look at (7.1) and (7.2) reveals that the only strictly positive payoff of the call option occurs in the state uu leading to a state-contingent payoff of,

$$\overline{C}_2(\omega) = \begin{cases} S_2^1(uu) - K & \text{for } \omega = uu \\ 0 & \text{for } \omega = um \\ 0 & \text{for } \omega = ud \\ 0 & \text{for } \omega = du \\ 0 & \text{for } \omega = dm \\ 0 & \text{for } \omega = dd \end{cases}.$$

The inspection of the replication sub-problem at the u node at $n = 1$,

$$\phi_2^0(u) \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \phi_2^1(u) \cdot \begin{pmatrix} \eta_2^{uu} \\ \eta_2^{um} \\ \eta_2^{ud} \end{pmatrix} \stackrel{!}{=} \begin{pmatrix} S_2^1(uu) - K \\ 0 \\ 0 \end{pmatrix}, \tag{7.10}$$

lets one conclude that it has no solution. This is because of the assumption,

$$\eta_2^{uu} > \eta_2^{um} > \eta_2^{ud}.$$

As a consequence, the call option $\overline{C}_2$ is not attainable.

Considering (7.3) and (7.4), one sees that the only strictly positive payoff of the put option $\overline{P}_2$ occurs in state dd,

$$\overline{P}_2(\omega) = \begin{cases} 0 & \text{for } \omega = uu \\ 0 & \text{for } \omega = um \\ 0 & \text{for } \omega = ud \\ 0 & \text{for } \omega = du \\ 0 & \text{for } \omega = dm \\ X - S_2^1(dd) & \text{for } \omega = dd \end{cases}.$$

The corresponding replication sub-problem at the d node at $n = 1$ for the put option is,

$$\phi_2^0(d) \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \phi_2^1(d) \cdot \begin{pmatrix} \eta_2^{du} \\ \eta_2^{dm} \\ \eta_2^{dd} \end{pmatrix} \stackrel{!}{=} \begin{pmatrix} 0 \\ 0 \\ X - S_2^1(dd) \end{pmatrix}. \tag{7.11}$$

It is easy to check that this sub-problem has no solution under the assumption,

$$\eta_2^{du} > \eta_2^{dm} > \eta_2^{dd}.$$

Thus, hedgers can neither perfectly hedge the call option $\overline{C}_2$ nor the put option $\overline{P}_2$. We will show, however, that they can super-replicate the options. As we will see shortly, this result is general insofar as every contingent claim is super-replicable in the market model $\mathcal{M}^{im}$. A proof of this claim is found in the subsequent section. The remainder of this sub-section is devoted to the derivation of the cost-minimizing super-replication strategies.

Call options

Suppose first that all hedgers hedge European call options of type $\overline{C}_2$, and define the aggregate target payoff of the hedgers by $\widetilde{C}_2 \equiv \alpha \cdot \overline{C}_2$. At the d node at $n = 1$, the hedgers face the replication sub-problem,

$$\phi_2^0(d) \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \phi_2^1(d) \cdot \begin{pmatrix} \eta_2^{du} \\ \eta_2^{dm} \\ \eta_2^{dd} \end{pmatrix} \stackrel{!}{=} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

As a result, they simply hold the 'zero portfolio', $\phi_2^0(d) = 0$ and $\phi_2^1(d) = 0$, there.

The hedgers' problem at the u node at $n = 1$ is not as trivial as at the d node. Since the replication sub-problem (7.10) has no solution, hedgers must super-replicate the call option at this node to ensure that they are able to honor their obligations from writing the call options. Formally, the hedgers face the problem of,

$$\begin{aligned} \min_{\phi_2^0(u),\phi_2^1(u)} \quad & \phi_2^0(u) + \phi_2^1(u) \cdot S_1^1(u) \\ \text{s.t. } & \phi_2^0(u) + \phi_2^1(u) \cdot S_2^1 \geq \widetilde{C}_2. \end{aligned}$$

More detailed, their problem is to,

$$\min_{\phi_2^0(u),\phi_2^1(u)} \quad \phi_2^0(u) + \phi_2^1(u) \cdot S_1^1(u) \tag{7.12}$$

$$\text{s.t. } \phi_2^0 + \phi_2^1 \cdot S_2^1(uu) \geq \widetilde{C}_2(uu) \tag{7.13}$$

$$\phi_2^0(u) + \phi_2^1(u) \cdot S_2^1(um) \geq 0 \tag{7.14}$$

$$\phi_2^0(u) + \phi_2^1(u) \cdot S_2^1(ud) \geq 0. \tag{7.15}$$

Problem (7.12)-(7.15) can easily be solved by graphic means. It seems convenient to conduct such an analysis in a diagram where we, as above,

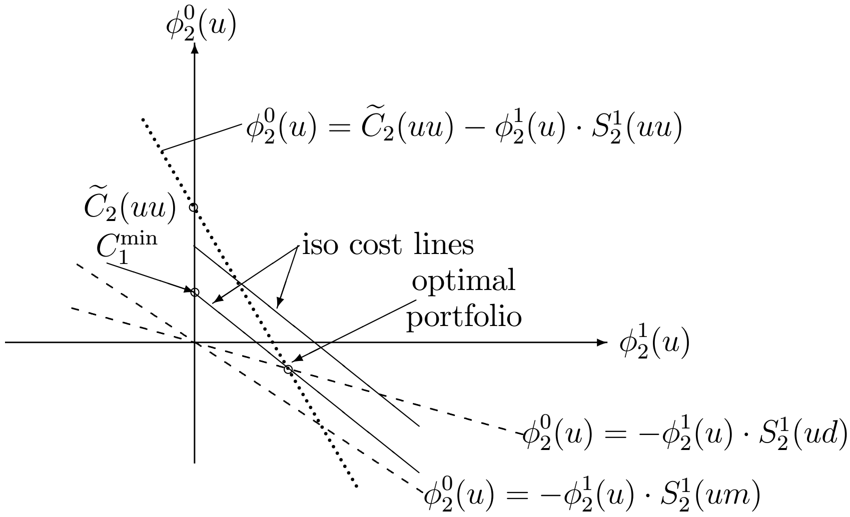

Figure 7.2: Minimizing super-replication costs for the call option.

put $\phi_2^0(u)$ on the vertical axis and $\phi_2^1(u)$ on the horizontal axis. Conditions (7.13)-(7.15) can be manipulated to obtain,

$$\phi_2^0(u) \geq \widetilde{C}_2(uu) - \phi_2^1(u) \cdot S_2^1(uu), \tag{7.16}$$

$$\phi_2^0(u) \geq -\phi_2^1(u) \cdot S_2^1(um), \tag{7.17}$$

$$\text{and } \phi_2^0(u) \geq -\phi_2^1(u) \cdot S_2^1(ud), \tag{7.18}$$

respectively. Now consider the objective function (7.12) and fix the hedge costs at the u node at date $n = 1$. The iso cost lines are then determined by,

$$\phi_2^0(u) = \widehat{C}_1(u) - \phi_2^1(u) \cdot S_1^1(u),$$

where $\widehat{C}_1(u)$ is fixed. It is now important to realize that by the absence of arbitrage,

$$S_2^1(uu) > S_1^1(u) > S_2^1(ud),$$

must hold since there is no arbitrage if and only if $S_1^1(u) \in]S_2^1(uu), S_2^1(ud)[$. This can be seen by considering the two possibilities $S_1^1(u) \geq S_2^1(uu)$ and $S_1^1(u) \leq S_2^1(ud)$. If $S_1^1(u) \geq S_2^1(uu)$, selling short the stock yields risk-less (expected) profits. Similarly, if $S_1^1(u) \leq S_2^1(ud)$, taking a long position in the stock locks in a risk-less (expected) profit.

Figure 7.2 illustrates the hedge problem (7.12)-(7.15). The intersection of,

$$\phi_2^0(u) = \widetilde{C}_2(uu) - \phi_2^1(u) \cdot S_2^1(uu),$$

and,

$$\phi_2^0(u) = -\phi_2^1(u) \cdot S_2^1(ud),$$

determines the optimal hedge portfolio while (7.17) is not binding in optimum. Therefore, the stock component can be derived from,

$$\begin{aligned}\widetilde{C}_2(uu) - \phi_2^1(u) \cdot S_2^1(uu) &= -\phi_2^1(u) \cdot S_2^1(ud),\\ \Leftrightarrow \phi_2^1(u) &= \frac{\widetilde{C}_2(uu)}{S_2^1(uu) - S_2^1(ud)}.\end{aligned}$$

This, in turn, gives rise to,

$$\phi_2^0(u) = -\frac{\widetilde{C}_2(uu)}{S_2^1(uu) - S_2^1(ud)} \cdot S_2^1(ud).$$

Correspondingly, the minimal hedge costs $C_1^{\min}$ at the u node are,

$$C_1^{\min}(u) = \frac{\widetilde{C}_2(uu)}{S_2^1(uu) - S_2^1(ud)} \cdot \left[S_1^1(u) - S_2^1(ud)\right].$$

The optimal portfolio generates a state-contingent payoff in states $\omega \in \{uu, um, ud\}$ of,

$$\begin{aligned}&\frac{\widetilde{C}_2(uu)}{S_2^1(uu) - S_2^1(ud)} \cdot \begin{pmatrix} S_2^1(uu) \\ S_2^1(um) \\ S_2^1(ud) \end{pmatrix}\\ &-\frac{\widetilde{C}_2(uu)}{S_2^1(uu) - S_2^1(ud)} \cdot S_2^1(ud) \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\\ = &\begin{pmatrix} \widetilde{C}_2(uu) \\ \widetilde{C}_2(uu) \cdot \frac{S_2^1(um) - S_2^1(ud)}{S_2^1(uu) - S_2^1(ud)} \\ 0 \end{pmatrix}.\end{aligned}$$

Here, it obviously holds that,

$$\frac{S_2^1(um) - S_2^1(ud)}{S_2^1(uu) - S_2^1(ud)} < 1,$$

so that the payoff in state uu exceeds the payoff in state um.[6]

[6] This property of state-contingent payoffs resulting from super-replication in discrete space settings is well-known. Compare example 4 in NAIK (1995).

The value of a call option at date $n = 1$, based on the super-replication argument, is $C_1^{\min}(u)$ at the u node and $C_1(d) = 0$ at the d node. The replication sub-problem at date $n = 0$,

$$\phi_1^0 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \phi_1^1 \cdot \begin{pmatrix} S_1^1(u) \\ S_1^1(d) \end{pmatrix} = \begin{pmatrix} C_1^{\min}(u) \\ 0 \end{pmatrix},$$

has a unique solution which is given by,

$$\phi_1^1 = \frac{C_1^{\min}(u)}{S_1^1(u) - S_1^1(d)},$$

and,

$$\phi_1^0 = -\frac{C_1^{\min}(u)}{S_1^1(u) - S_1^1(d)} \cdot S_1^1(d).$$

From these expressions, the minimal initial costs to implement the super-replicating strategy can be derived as,

$$C_0^{\min} = \frac{C_1^{\min}(u)}{S_1^1(u) - S_1^1(d)} \cdot \left[S_0^1 - S_1^1(d)\right].$$

To conclude the analysis of call option super-replication, we want to note that the profit or loss of the hedgers in this case is given as,

$$\pi \equiv \alpha \cdot S_0^1 - C_0^{\min}.$$

Put options

Now suppose that all hedgers hedge European put options of type $\overline{P}_2$ and let $\widetilde{P}_2 \equiv \alpha \cdot \overline{P}_2$ denote the aggregate target payoff at $n = 2$. When hedging puts, it is optimal for the hedgers to hold a zero portfolio at the u node at $n = 1$, $\phi_2^0(u) = 0$ and $\phi_2^1(u) = 0$. This becomes obvious by considering the corresponding sub-problem,

$$\phi_2^0(u) \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \phi_2^1(u) \cdot \begin{pmatrix} \eta_2^{uu} \\ \eta_2^{um} \\ \eta_2^{ud} \end{pmatrix} \stackrel{!}{=} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

We found above that the sub-problem (7.11) at the d node at $n = 1$ has no solution. As a result, the hedgers must super-replicate the put options at this node. This is the only strategy that enables the hedgers to meet

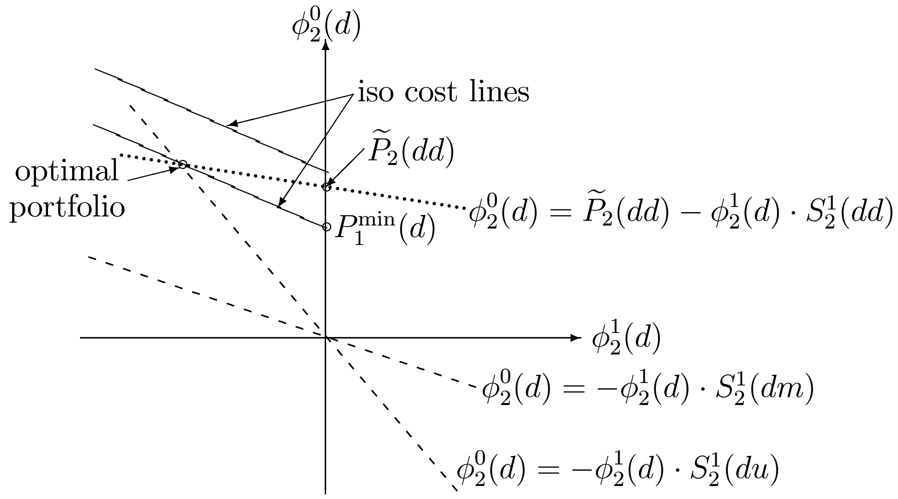

Figure 7.3: Minimizing super-replication costs for the put option.

the obligations from the written options with certainty. The minimization problem the hedgers have to solve is,

$$\min_{\phi_2^0(d),\phi_2^1(d)} \phi_2^0(d) + \phi_2^1(d) \cdot S_1^1(d) \tag{7.19}$$

$$\text{s.t. } \phi_2^0(d) + \phi_2^1(d) \cdot S_2^1 \geq \widetilde{P}_2. \tag{7.20}$$

A graphic analysis will again provide the solution to this problem quickly. Condition (7.20) translates into the three constraints,

$$\phi_2^0(d) \geq -\phi_2^1(d) \cdot S_2^1(du) \tag{7.21}$$

$$\phi_2^0(d) \geq -\phi_2^1(d) \cdot S_2^1(dm), \tag{7.22}$$

$$\phi_2^0(d) \geq \widetilde{P}_2(dd) - \phi_2^1(d) \cdot S_2^1(dd). \tag{7.23}$$

For fixed $\widehat{P}_1(d)$, the iso cost lines are determined by,

$$\phi_2^0(d) = \widehat{P}_1(d) - \phi_2^1(d) \cdot S_1^1(d).$$

Similar arbitrage reasoning as in the call case reveals that there is no arbitrage if and only if $S_1^1(d) \in]S_2^1(du), S_2^1(dd)[$.

Problem (7.19) and (7.20) is depicted in figure 7.3. Apparently, the intersection of,

$$\phi_2^0(d) = -\phi_2^1(d) \cdot S_2^1(du),$$

and,

$$\phi_2^0(d) = \widetilde{P}_2(dd) - \phi_2^1(d) \cdot S_2^1(dd),$$

determines the optimal hedge portfolio. Using this information, simple calculations yield,

$$\phi_2^1(d) = -\frac{\widetilde{P}_2(dd)}{S_2^1(du) - S_2^1(dd)}, \tag{7.24}$$

$$\phi_2^0(d) = \frac{\widetilde{P}_2(dd)}{S_2^1(du) - S_2^1(dd)} \cdot S_2^1(du), \tag{7.25}$$

as the optimal portfolio. The corresponding minimal hedge costs $P_1^{\min}(d)$ at date $n = 1$ are,

$$P_1^{\min}(d) = \frac{\widetilde{P}_2(dd)}{S_2^1(du) - S_2^1(dd)} \cdot \left[S_2^1(du) - S_1^1(d)\right].$$

Given the optimal super-replication portfolio (7.24) and (7.25), the state-contingent payoff that the hedgers actually achieve in states $\omega \in \{du, dm, dd\}$ is,

$$\begin{pmatrix} 0 \\ \widetilde{P}_2(dd) \cdot \frac{S_2^1(du) - S_2^1(dm)}{S_2^1(du) - S_2^1(dd)} \\ \widetilde{P}_2(dd) \end{pmatrix}.$$

By assumption,

$$\frac{S_2^1(du) - S_2^1(dm)}{S_2^1(du) - S_2^1(dd)} < 1,$$

so that the payoff in state dm amounts to less than the payoff in dd.

Solving the replication sub-problem at date $n = 0$,

$$\phi_1^0 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \phi_1^1 \cdot \begin{pmatrix} S_1^1(u) \\ S_1^1(d) \end{pmatrix} = \begin{pmatrix} 0 \\ P_1^{\min}(d) \end{pmatrix},$$

is now an easy exercise. The unique solution is,

$$\phi_1^1 = -\frac{P_1^{\min}(d)}{S_1^1(u) - S_1^1(d)},$$

$$\phi_1^0 = \frac{P_1^{\min}(d)}{S_1^1(u) - S_1^1(d)} \cdot S_1^1(u),$$

yielding minimal initial costs of,

$$P_0^{\min} = \frac{P_1^{\min}(d)}{S_1^1(u) - S_1^1(d)} \cdot \left[S_1^1(u) - S_0^1\right].$$

The profit or loss of the hedgers exclusively hedging puts is,

$$\pi \equiv \alpha \cdot S_0^1 - P_0^{\min}.$$

This concludes the characterization of the optimal super-replication strategies for the call option $\overline{C}_2$ and the put option $\overline{P}_2$. In summary, super-replication of call options of type $\overline{C}_2$ and put options $\overline{P}_2$ creates an actual state-contingent payoff per option of,

$$\begin{cases} \overline{C}_2(uu) & \text{for } \omega = uu \\ \overline{C}_2(uu) \cdot \frac{S_2^1(um) - S_2^1(ud)}{S_2^1(uu) - S_2^1(ud)} & \text{for } \omega = um \\ 0 & \text{for } \omega = ud \\ 0 & \text{for } \omega = du \\ 0 & \text{for } \omega = dm \\ 0 & \text{for } \omega = dd \end{cases}, \tag{7.26}$$

in the former case and,

$$\begin{cases} 0 & \text{for } \omega = uu \\ 0 & \text{for } \omega = um \\ 0 & \text{for } \omega = ud \\ 0 & \text{for } \omega = du \\ \overline{P}_2(dd) \cdot \frac{S_2^1(du) - S_2^1(dm)}{S_2^1(du) - S_2^1(dd)} & \text{for } \omega = dm \\ \overline{P}_2(dd) & \text{for } \omega = dd \end{cases}, \tag{7.27}$$

in the latter case. These are the contingent claims we actually have to take into account in the analysis to follow. Super-replication of both the call and the put yields actual state-contingent payoffs that are unique and, even more important for our purposes, independent of the stock prices prior to $n = 2$.

7.4 Equilibrium analysis

This section investigates the existence and determinacy of general equilibria in the market model $\mathcal{M}^{im}$. The exposition is very concise since there is hardly a difference in comparison to the analysis in section 6.3. The main reason for this is the existence of a representative non-hedger who sets prices in equilibrium. Fortunately, the analysis of such an agent's problem is very similar in both complete and incomplete markets setting. In this section, we also demonstrate that the market model $\mathcal{M}^{im}$ is incomplete by construction. Every contingent claim is, however, super-replicable as we will argue.

We have as the counterpart to Theorem 49,

Theorem 64 *Assume that the hedgers hedge calls of type* (7.1) *and* (7.2) *and puts of type* (7.3) *and* (7.4) *only. Then there exists a unique general equilibrium for the market model,*

$$\mathcal{M}^{im} = \{(\Omega, \wp(\Omega), \mathbb{F}, \mathbf{P}), N = 2, \mathbb{S}^1, \mathbb{I}^\alpha\},$$

as defined in definition 48. The equilibrium stock prices satisfy,

$$S_n^1 = \frac{\mathbf{E}_n^{\mathbf{P}}\left[v'\left(\eta_2 - A_2\right) \cdot \eta_2\right]}{\mathbf{E}_n^{\mathbf{P}}\left[v'\left(\eta_2 - A_2\right)\right]}, \tag{7.28}$$

for $n \in \{0, 1\}$, *and* $S_2^1 = \eta_2$ *for* $n = 2$. $\mathbf{E}_n^{\mathbf{P}}$ *is the conditional expectation given the information set* $\mathcal{F}_n$, $v'(\cdot)$ *is the first derivative of the non-hedgers' utility function* $v(\cdot)$, η_2 *is the liquidating dividend of the stock at* $n = 2$ *and* A_2 *is the actual* $n = 2$ *state-contingent payoff that the hedgers achieve.*

Proof. Due to A_2 being unique and independent of the equilibrium stock prices at $n = 0$ and $n = 1$, the proof of this Theorem closely parallels the proof of Theorem 49. The only exception constitutes the fact that one has to deal with six instead of four possible states at the terminal date. To see this, recall that the proof of Theorem 49 was given before we knew that the market model $\mathcal{M}^{cm}$ is complete. Therefore, we had to carry out the proof in a rather general fashion to take into account the possibility of incomplete markets. In particular, we worked with a martingale basis because we could not know whether $\mathbb{Q}$ is a singleton or not. The same methodology perfectly applies here so that we can omit a detailed proof of Theorem 64. ■

Results of Duffie and Huang (1985) allow a short proof of the following claim.

Lemma 65 *The market model* $\mathcal{M}^{im} = \{(\Omega, \wp(\Omega), \mathbb{F}, \mathbf{P}), N = 2, \mathbb{S}^1, \mathbb{I}^\alpha\}$ *is incomplete.*

Proof. As seen above, the present market model can be represented by an event tree. In the terminology of Duffie and Huang (1985), the market model includes so-called *long-lived* securities. In our setting, this means that such securities are traded at all dates $n \in \{0, 1, 2\}$. Provided there are no other securities available, Duffie and Huang (1985) identify in sub-section 6.3 as a necessary condition for such a market model to be complete the following: *There have to be at least as many long-lived securities available as the maximum number of branches leaving any node of the tree.* The present market model, however, contains only two long-lived securities whereas the maximum number of branches leaving any node is three. Due to this mismatch, the market model is incomplete. ■

Even though there is a unique general equilibrium, the market model $\mathcal{M}^{im}$ is incomplete. This is in contrast to the market model $\mathcal{M}^{cm}$ of the last chapter where uniqueness implied completeness under the common knowledge assumption. Yet the market structure is regular enough for all contingent claims to be super-replicable.

Lemma 66 *In the market model* $\mathcal{M}^{im} = \{(\Omega, \wp(\Omega), \mathbb{F}, \mathbf{P}), N = 2, \mathbb{S}^1, \mathbb{I}^\alpha\}$, *every contingent claim is super-replicable, or equivalently,* $\mathbb{A}^* = \mathbb{R}^6_+$.

Proof. It suffices to identify *one* admissible trading strategy that super-replicates an arbitrary contingent claim $A_2 \in \mathbb{R}^6_+$. One such strategy is, for instance, $(\phi_n)_{n\in\{0,1,2\}} \in \mathbb{T}$ where,

$$\begin{aligned} \forall n &\in \{0,1,2\} : (\phi_n^0, \phi_n^1) = (a, 0), \\ a &\equiv \max\{A_2(\omega) : \omega \in \Omega\}. \end{aligned}$$

Exclusively investing in the bond, this strategy generates a state-independent payoff at $n = 2$. The payoff in all states $\omega \in \Omega$ equals the maximum of what the contingent claim pays off in any state ω. This guarantees that the payoff of the admissible trading strategy $V_2(\phi)$ dominates the payoff of the contingent claim A_2 in all states, $\forall \omega \in \Omega : V_2(\phi) \geq A_2$. ■

Although the market model $\mathcal{M}^{im}$ is incomplete in the sense of definition 34, the market *sub-model* between dates $n = 0$ and $n = 1$ is complete in the sense that every payoff $A_1 \in \mathbb{R}^2_+$ at $n = 1$ can be created via trading in the stock and the bond at $n = 0$.

Lemma 67 *Under the assumptions of Theorem 64, the market sub-model between dates* $n = 0$ *and* $n = 1$ *of* $\mathcal{M}^{im} = \{(\Omega, \wp(\Omega), \mathbb{F}, \mathbf{P}), N = 2, \mathbb{S}^1, \mathbb{I}^\alpha\}$ *is complete.*

Proof. Given an arbitrary $n = 1$ payoff $A_1 \in \mathbb{R}^2_+$, the lemma directly follows from Theorem 64 since the linear system,

$$\phi_1^0 + \phi_1^1 \cdot S_1^1 = A_1$$
$$\Rightarrow \left\{ \begin{array}{l} \phi_1^0 + \phi_1^1 \cdot S_1^1(u) = A_1(u) \\ \phi_1^0 + \phi_1^1 \cdot S_1^1(d) = A_1(d) \end{array} \right. ,$$

has a unique solution in ϕ_1^0 and ϕ_1^1 for all $A_1 \in \mathbb{R}^2_+$ and all (unique) equilibrium stock prices $S_1^1(u)$ and $S_1^1(d)$. ■

As before, we declare the case $\alpha = 0$ as the benchmark case and label expressions related to this particular case with the superscript "*".

We are now prepared to proceed in the subsequent section with the comparative statics analysis of dynamic hedging.

7.5 Comparative statics analysis

This section carries out the comparative statics analysis. On the one hand, it considers settings where hedgers only hedge calls, and on the other hand, settings where hedgers simultaneously hedge calls and puts. In combination with the assumption that there are only two nodes at $n = 1$, this enables us to directly compare the results with those obtained in section 6.5.

Assume that every hedger dynamically hedges either one European call option given by (7.1) and (7.2) or one European put option given by (7.3) and (7.4). To be consistent with the analysis in section 6.5, we again examine two different cases. Assuming that a fraction $\rho \in [0, 1]$ of α hedges call options of type $\overline{C}_2$ and a fraction $1 - \rho$ hedges put options of type $\overline{P}_2$, we explore in sub-sections 7.5.1 and 7.5.2 the two parameter settings

(a) $\rho = 1$ and

(b) $\rho \in [0, 1]$,

respectively.

7.5.1 Dynamic hedging of calls

In this (and the next) sub-section, we will see that we are not able to reproduce the comparative statics results of the last chapter. There we found that dynamic hedging of calls unambiguously increased the stock price $S_1^1(u)$ at the u node at $n = 1$. As an immediate consequence, the volatility σ rose as well.

Consider case (a) where hedgers only hedge calls of type $\overline{C}_2$. In the present context where hedgers must super-replicate the call option $\overline{C}_2$, the stock price may either increase, decrease or remain unchanged in the presence of hedgers. This is due to the fact that super-replication smooths the state-contingent payoff the hedgers receive at the terminal date [see (7.26)]. We have,

Proposition 68 *In the market model $\mathcal{M}^{im}$, the equilibrium stock price $S_1^1(u)$ at the u node at date $n = 1$ either increases, decreases or stays the same when the market weight α of hedgers super-replicating calls of type $\overline{C}_2$ increases.*

Proof. Sub-section 7.8.1 contains the proof. ■

Proposition 68 implies that the net effect of dynamic hedging on the volatility is ambiguous as well.

Corollary 69 *Volatility σ may increase, decrease or remain unchanged when the market weight α of hedgers super-replicating calls of type $\overline{C}_2$ increases.*

Proof. Recalling the definition of volatility σ and noting that by (7.26) and (7.28) $S_1^1(d) = S_1^{1*}(d)$, one obtains from proposition 68,

$$\begin{aligned}
\frac{\partial \sigma}{\partial \alpha} &= \frac{\partial}{\partial \alpha}\left(S_1^1(u) - S_1^1(d)\right) \\
&= \frac{\partial}{\partial \alpha}\left(S_1^1(u) - S_1^{1*}(d)\right) \\
&= \frac{\partial}{\partial \alpha} S_1^1(u) - \underbrace{\frac{\partial}{\partial \alpha} S_1^{1*}(d)}_{=0} \\
&\lesseqgtr 0,
\end{aligned}$$

as asserted. ■

Moreover, the question of how volatility changes relative to the benchmark case in the presence of dynamic hedging can not be answered satisfactorily.

Corollary 70 *When hedgers with non-zero market weight α super-replicate calls of type $\overline{C}_2$, volatility σ may satisfy,*

$$\sigma \lesseqgtr \sigma^*,$$

where σ^* denotes the volatility in the benchmark case $\alpha = 0$.

Proof. Clear. ■

Market incompleteness requires hedgers to super-replicate the given call option. The state-contingent payoff that results from super-replication is characterized by strictly positive values in states uu and um while the call option itself has a strictly positive payoff in the uu state only. This factor results in a smoothing effect. Due to this effect, it is no longer clear what impact dynamic hedging of call options has on the stock price $S_1^1(u)$ and on volatility σ. In order to arrive at unambiguous results, we would have to specify almost all parameters in the market model $\mathcal{M}^{im}$.

7.5.2 Dynamic hedging of calls and puts

Consider now case (b) where a fraction $\rho \in [0, 1]$ of α hedges calls of type $\overline{C}_2$ and where the remaining fraction $1 - \rho$ hedges put options of type $\overline{P}_2$. Under these assumptions, the aggregate state-contingent payoff that a single hedger on average achieves by following super-replication schemes is,

$$\begin{aligned}
\overline{A}_2 \quad &\equiv \quad \rho \cdot \begin{pmatrix} \overline{C}_2(uu) \\ \overline{C}_2(uu) \cdot \frac{S_2^1(um) - S_2^1(ud)}{S_2^1(uu) - S_2^1(ud)} \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \\
&\qquad + (1-\rho) \cdot \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \overline{P}_2(dd) \cdot \frac{S_2^1(du) - S_2^1(dm)}{S_2^1(du) - S_2^1(dd)} \\ \overline{P}_2(dd) \end{pmatrix} \\
&= \begin{pmatrix} \rho \cdot \overline{C}_2(uu) \\ \rho \cdot \overline{C}_2(uu) \cdot \frac{S_2^1(um) - S_2^1(ud)}{S_2^1(uu) - S_2^1(ud)} \\ 0 \\ 0 \\ (1-\rho) \cdot \overline{P}_2(dd) \cdot \frac{S_2^1(du) - S_2^1(dm)}{S_2^1(du) - S_2^1(dd)} \\ (1-\rho) \cdot \overline{P}_2(dd) \end{pmatrix}
\end{aligned} \tag{7.29}$$

Denote by,

$$A_2 \equiv \alpha \cdot \overline{A}_2, \tag{7.30}$$

the actual aggregate state-contingent payoff that the hedgers achieve.

In terms of intuition, we have little to add to the things already pointed out in the previous sub-section. For reasons of completeness, however, we list the results corresponding to those of the last sub-section below. An interesting special case included in the present setting is $\rho = 0$ where hedgers exclusively hedge puts. This case resembles the portfolio insurance economies considered in Brennan and Schwartz (1989), Basak (1995) or Grossman and Zhou (1996). In contrast to sub-section 6.5.2, our findings in this sub-section show that dynamic put hedging, or equivalently, portfolio insurance may *both* increase volatility - as found in Brennan and Schwartz (1989) and Grossman and Zhou (1996) - and decrease volatility - as found in Basak (1995).

Proposition 71 *In the market model $\mathcal{M}^{im}$, regardless of the actual value $\rho \in [0,1]$, equilibrium stock prices $S_1^1(u)$ and $S_1^1(d)$ at date $n = 1$ may both independently increase, decrease or remain unchanged when the market weight α of hedgers super-replicating calls of type $\overline{C}_2$ and puts of type $\overline{P}_2$ increases.*

Proof. Sub-section 7.8.2 contains the proof. ■

Corollary 72 *Volatility σ may increase, decrease or remain unchanged when the market weight α of hedgers super-replicating calls of type $\overline{C}_2$ and puts of type $\overline{P}_2$ increases.*

Proof. Clear. ■

Corollary 73 *Volatility σ may satisfy,*

$$\sigma \lesseqgtr \sigma^*,$$

when hedgers with non-zero market weight α super-replicate calls of type $\overline{C}_2$ and puts of type $\overline{P}_2$. σ^* denotes the volatility in the benchmark case $\alpha = 0$.

Proof. Clear. ■

In the next section, numerical examples are provided which illustrate the impact of dynamic hedging on financial market equilibria.

7.6 Numerical computations

This section provides a number of numerical computations for European call and put options. It contains computations for the market models $\mathcal{M}^{cm}$ and $\mathcal{M}^{im}$. For concreteness, consider a setting where,

$$\eta_2 \in \left\{ \begin{array}{c} \eta_2^{uu} = 120 \\ (\eta_2^{um} = 110) \\ \eta_2^{ud} = 100 \\ \eta_2^{du} = 100 \\ (\eta_2^{dm} = 90) \\ \eta_2^{dd} = 80 \end{array} \right. .$$

Of course, states um and dm only exist in the market model $\mathcal{M}^{im}$. That is why we put them in parentheses. Every state is equally likely to unfold. In the market model $\mathcal{M}^{cm}$, this means that $\forall \omega \in \Omega : \mathbf{P}(\omega) = \frac{1}{4}$ while it means in the market model $\mathcal{M}^{im}$ that $\forall \omega \in \Omega : \mathbf{P}(\omega) = \frac{1}{6}$. Consider now the European call option given by,

$$C_2 : \mathbb{R}_{++} \to \mathbb{R}_+, S_2^1 \mapsto \max\{S_2^1 - 110, 0\}, \tag{7.31}$$

and the European put option given by,

$$P_2 : \mathbb{R}_{++} \to \mathbb{R}_+, S_2^1 \mapsto \max\{90 - S_2^1, 0\}. \tag{7.32}$$

Obviously, these options fit into the classes of options as described by (6.33), (6.34), (6.35) and (6.36) on the one hand and (7.1), (7.2), (7.3) and (7.4) on the other hand. To derive stock prices according to the pricing equations (6.18) and (7.28), we additionally assume that the non-hedgers exhibit CRRA, i.e., they have a utility function,

$$v : \mathbb{R}_+ \to \mathbb{R}, w \mapsto \frac{w^{1-\gamma}}{1-\gamma},$$

where γ is the constant degree R_r of relative risk aversion.[7]

Tables 7.1 and 7.2 contain the data for the market model $\mathcal{M}^{cm}$. Volatility figures in all tables are to be understood relative to the case where $\alpha = 0$ and $\gamma = 1$. From tables 7.1 and 7.2 we can conclude that volatility increases with both the market weight α of hedgers and the degree of CRRA. This is consistent with results reported in Brennan and Schwartz (1989) as well as Balduzzi, Bertola, and Foresi (1995).

Table 7.1: Volatility in the market model $\mathcal{M}^{cm}$ when the call option (7.31) is hedged.

γ \ α	0.00	0.05	0.10	0.15	0.20	0.25
1	100.00	100.10	100.21	100.31	100.41	100.52
2	100.94	101.14	101.34	101.54	101.75	101.95
3	101.77	102.06	102.35	102.64	102.93	103.23
4	102.44	102.81	103.17	103.55	103.92	104.30

Table 7.2: Volatility in the market model $\mathcal{M}^{cm}$ when the put option (7.32) is hedged.

γ \ α	0.00	0.05	0.10	0.15	0.20	0.25
1	100.00	100.15	100.31	100.46	100.62	100.77
2	100.94	101.23	101.53	101.83	102.13	102.42
3	101.77	102.18	102.60	103.01	103.43	103.84
4	102.44	102.95	103.46	103.96	104.46	104.96

As seen above, options like (7.31) and (7.32) cannot be perfectly replicated in the market model $\mathcal{M}^{im}$. We nevertheless consider the hypothetical case

[7] Refer to section 3.3.

that the hedgers realize a state-contingent payoff per capita equal to the call or the put. This enables the separation of two distinct effects when exchanging the market models $\mathcal{M}^{cm}$ and $\mathcal{M}^{im}$. The first effect is the one induced by the change in the market structure itself (e.g., a change in the probability measure $\mathbf{P}$). The second effect is the one induced by the necessity to super-replicate the options. Super-replication leads, as demonstrated, to a smoother state-contingent payoff. Data for the hypothetical case is provided in tables 7.3 and 7.4. A comparison with tables 7.1 and 7.2 reveals that the observed effects in the market model $\mathcal{M}^{cm}$ are more pronounced than in the market model $\mathcal{M}^{im}$. Mathematically, this is mainly due to the fact that in both cases the only state where the hedgers have a strictly positive demand has more weight in $\mathcal{M}^{cm}$ (where $\forall\omega \in \Omega : \mathbf{P}(\omega) = \frac{1}{4}$) than in $\mathcal{M}^{im}$ (where $\forall\omega \in \Omega : \mathbf{P}(\omega) = \frac{1}{6}$).

Table 7.3: Volatility in the market model $\mathcal{M}^{im}$ if the call option (7.31) could be perfectly hedged.

γ \ α	0.00	0.05	0.10	0.15	0.20	0.25
1	100.00	100.07	100.13	100.20	100.27	100.34
2	100.66	100.79	100.92	101.05	101.18	101.31
3	101.29	101.47	101.66	101.84	102.03	102.22
4	101.87	102.10	102.33	102.56	102.80	103.05

Table 7.4: Volatility in the market model $\mathcal{M}^{im}$ if the put option (7.32) could be perfectly hedged.

γ \ α	0.00	0.05	0.10	0.15	0.20	0.25
1	100.00	100.11	100.22	100.32	100.43	100.54
2	100.66	100.88	101.10	101.32	101.55	101.77
3	101.29	101.62	101.95	102.28	102.62	102.96
4	101.87	102.30	102.74	103.18	103.62	104.07

If we now turn to the more realistic case where the options are super-replicated (tables 7.5 and 7.6), we see that the necessity for super-replication has a smoothing effect, too. The increase in volatility is not as sharp as in $\mathcal{M}^{cm}$ or as in the hypothetical case in $\mathcal{M}^{im}$. As a matter of fact, super-replication leads to a state-contingent payoff that is smoother than the state-contingent payoff of the options themselves. Consequently, super-replication affects the volatility of the stock less than (fictitious) perfect replication.

Table 7.5: Volatility in the market model $\mathcal{M}^{im}$ when the call option (7.31) is super-replicated.

γ \ α	0.00	0.05	0.10	0.15	0.20	0.25
1	100.00	100.07	100.14	100.21	100.28	100.35
2	100.66	100.80	100.94	101.07	101.21	101.35
3	101.29	101.49	101.70	101.90	102.11	102.31
4	101.87	102.13	102.39	102.66	102.93	103.20

Table 7.6: Volatility in the market model $\mathcal{M}^{im}$ when the put option (7.32) is super-replicated.

γ \ α	0.00	0.05	0.10	0.15	0.20	0.25
1	100.00	100.10	100.21	100.31	100.42	100.53
2	100.66	100.87	101.07	101.28	101.49	101.70
3	101.29	101.59	101.89	102.19	102.49	102.80
4	101.87	102.25	102.63	103.02	103.41	103.79

Figure 7.4 illustrates these statements for $\gamma = 1$. The increase in volatility is more pronounced in $\mathcal{M}^{cm}$ than in the hypothetical case in $\mathcal{M}^{im}$. In turn, the increase there is more pronounced than in the super-replication case in $\mathcal{M}^{im}$. The figure suggests that the effect due to the change in the market structure is much stronger than the one due to super-replication.

7.7 Summary

This chapter investigated dynamic hedging in a general equilibrium framework where markets are incomplete by construction. We showed the existence and uniqueness of a general equilibrium under our assumptions. This was possible because of the existence of a representative non-hedger and the restriction to certain contingent claims.

In incomplete markets settings, hedgers cannot achieve a perfect hedge in general. However, the market model laid out in this chapter offered the possibility to super-replicate every contingent claim. This ensured that the hedgers could at least super-replicate contingent claims in cases where a perfect hedge was infeasible. In such a case one usually speaks of a *complete* hedge as opposed to a *perfect* hedge.

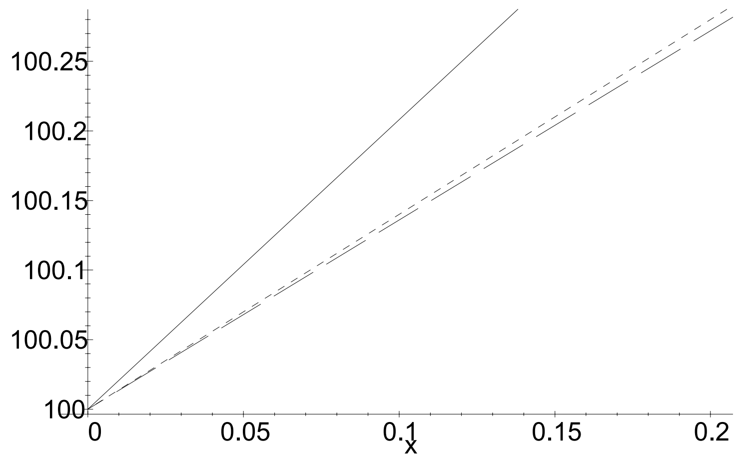

Figure 7.4: Volatility in $\mathcal{M}^{cm}$ [solid line] and in $\mathcal{M}^{im}$ [hypothetical: dashed line / super-replication: dotted line] for $\gamma = 1$. x denotes α.

The comparative statics analysis focused on European call and put options. Our main motivation for this decision was that it allows us to compare results obtained in chapter 6 with those obtained in this chapter. As it turned out, it is no longer clear how dynamic hedging influences markets where it is implemented, even in cases where it was completely clear, such as in chapter 6. While we found in chapter 6 that dynamic hedging of call and put options unambiguously increases stock price volatility, we found here that it may also decrease volatility. This was primarily due to the fact that super-replication generates smoother state-contingent payoffs than the actual state-contingent payoffs of the options. Numerical computations illustrated some of the analytical results.

The findings of this chapter support our arguments made in chapter 6. Dynamic hedging of contingent claims with convex payoffs does not necessarily lead to a rise in stock price volatility. Apart from this conclusion, this chapter also contributes to the literature about dynamic hedging in imperfectly liquid markets in that it considers a market model that is incomplete a priori. It would be interesting to explore dynamic hedging in such a context when the hedgers follow an alternative hedging scheme or when the hedgers hedge more general contingent claims.

7.8 Mathematical proofs

7.8.1 Proof of proposition 68

First note that $S_1^1(d) = S_1^{1*}(d)$ holds since there is no strictly positive hedge demand in the lower part of the tree, i.e., for $\omega \in \{du, dm, dd\}$. By definition,

$$\begin{aligned}\sigma &= S_1^1(u) - S_1^1(d) \\ &= S_1^1(u) - S_1^{1*}(d),\end{aligned}$$

so that we only have to prove that,

$$\frac{\partial S_1^1(u)}{\partial \alpha} \lesseqgtr 0.$$

Denote by $\overline{A}_2$ the state-contingent payoff that the super-replication strategy for the call option $\overline{C}_2$ produces [see (7.26)]. Recall that by definition $\overline{A}_2 \geq \overline{C}_2$. Furthermore, denote $A_2 \equiv \alpha \cdot \overline{A}_2$.

$S_1^1(u)$ is given by,

$$S_1^1(u) = \frac{\mathbf{E}_1^{\mathbf{P}}[v'(\eta_2 - \alpha \cdot \overline{A}_2) \cdot \eta_2]}{\mathbf{E}_1^{\mathbf{P}}[v'(\eta_2 - \alpha \cdot \overline{A}_2)]},$$

where,

$$\begin{aligned}\mathbf{E}_1^{\mathbf{P}}[v'(\eta_2 - \alpha \cdot \overline{A}_2) \cdot \eta_2] \quad = \quad & \frac{1}{\mathbf{P}(uu) + \mathbf{P}(um) + \mathbf{P}(ud)} \\ & \cdot \left[\mathbf{P}(uu) \cdot v'(\eta_2^{uu} - \alpha \cdot \overline{A}_2(uu)) \cdot \eta_2^{uu} \right. \\ & +\mathbf{P}(um) \cdot v'(\eta_2^{um} - \alpha \cdot \overline{A}_2(um)) \cdot \eta_2^{um} \\ & \left. +\mathbf{P}(ud) \cdot v'(\eta_2^{ud}) \cdot \eta_2^{ud}\right],\end{aligned}$$

and,

$$\begin{aligned}\mathbf{E}_1^{\mathbf{P}}[v'(\eta_2 - \alpha \cdot \overline{A}_2)] \quad = \quad & \frac{1}{\mathbf{P}(uu) + \mathbf{P}(um) + \mathbf{P}(ud)} \\ & \cdot \left[\mathbf{P}(uu) \cdot v'(\eta_2^{uu} - \alpha \cdot \overline{A}_2(uu)) \right. \\ & +\mathbf{P}(um) \cdot v'(\eta_2^{um} - \alpha \cdot \overline{A}_2(um)) \\ & \left. +\mathbf{P}(ud) \cdot v'(\eta_2^{ud})\right].\end{aligned}$$

Define the single conditional probabilities by,

$$\forall \omega \in \{uu, um, ud\} : \mathbf{P}^c(\omega) \equiv \frac{\mathbf{P}(\omega)}{\mathbf{P}(uu) + \mathbf{P}(um) + \mathbf{P}(ud)}.$$

One obtains,

$$
\begin{aligned}
&\frac{\partial S_1^1(u)}{\partial \alpha} \\
= \quad & \underbrace{\frac{\mathbf{P}^c(uu)\cdot v''(\eta_2^{uu}-\alpha\cdot\overline{A}_2(uu))\cdot\eta_2^{uu}\cdot(-\overline{A}_2(uu))\cdot\mathbf{E}_1^{\mathbf{P}}[v'(\eta_2-A_2)]}{\mathbf{E}_1^{\mathbf{P}}[v'(\eta_2-A_2)]^2}}_{(a)} \\
& + \underbrace{\frac{\mathbf{P}^c(um)\cdot v''(\eta_2^{um}-\alpha\cdot\overline{A}_2(um))\cdot\eta_2^{um}\cdot(-\overline{A}_2(um))\cdot\mathbf{E}_1^{\mathbf{P}}[v'(\eta_2-A_2)]}{\mathbf{E}_1^{\mathbf{P}}[v'(\eta_2-A_2)]^2}}_{(b)} \\
& - \underbrace{\frac{\mathbf{P}^c(uu)\cdot v''(\eta_2^{uu}-\alpha\cdot\overline{A}_2(uu))\cdot(-\overline{A}_2(uu))\cdot\mathbf{E}_1^{\mathbf{P}}[v'(\eta_2-A_2)\cdot\eta_2]}{\mathbf{E}_1^{\mathbf{P}}[v'(\eta_2-A_2)]^2}}_{(c)} \\
& - \underbrace{\frac{\mathbf{P}^c(um)\cdot v''(\eta_2^{um}-\alpha\cdot\overline{A}_2(um))\cdot(-\overline{A}_2(um))\cdot\mathbf{E}_1^{\mathbf{P}}[v'(\eta_2-A_2)\cdot\eta_2]}{\mathbf{E}_1^{\mathbf{P}}[v'(\eta_2-A_2)]^2}}_{(d)}
\end{aligned}
$$

We now demonstrate that $(a)-(c)>0$ (step 1) and that $(b)-(c)\lesseqgtr 0$ (step 2) which leads us to the final conclusion that $(a)+(b)-(c)-(d)\lesseqgtr 0$, or equivalently, $\frac{\partial S_1^1(u)}{\partial\alpha}\lesseqgtr 0$ (step 3).

Step 1: $(a)-(c)$ yields,

$$
\begin{aligned}
& \frac{\mathbf{P}^c(uu)\cdot v''(\eta_2^{uu}-\alpha\cdot\overline{A}_2(uu))\cdot\eta_2^{uu}\cdot(-\overline{A}_2(uu))\cdot\mathbf{E}_1^{\mathbf{P}}[v'(\eta_2-A_2)]}{\mathbf{E}_1^{\mathbf{P}}[v'(\eta_2-A_2)]^2} \\
& -\frac{\mathbf{P}^c(uu)\cdot v''(\eta_2^{uu}-\alpha\cdot\overline{A}_2(uu))\cdot(-\overline{A}_2(uu))\cdot\mathbf{E}_1^{\mathbf{P}}[v'(\eta_2-A_2)\cdot\eta_2]}{\mathbf{E}_1^{\mathbf{P}}[v'(\eta_2-A_2)]^2} \\
& \qquad = \frac{\mathbf{P}^c(uu)\cdot v''(\eta_2^{uu}-\alpha\cdot\overline{A}_2(uu))\cdot(-\overline{A}_2(uu))}{\mathbf{E}_1^{\mathbf{P}}[v'(\eta_2-A_2)]^2} \\
& \qquad \cdot\big[\eta_2^{uu}\cdot\mathbf{E}_1^{\mathbf{P}}[v'(\eta_2-A_2)]-\mathbf{E}_1^{\mathbf{P}}[v'(\eta_2-A_2)\cdot\eta_2]\big] \\
& \qquad = \frac{\mathbf{P}^c(uu)\cdot v''(\eta_2^{uu}-\alpha\cdot\overline{A}_2(uu))\cdot(-\overline{A}_2(uu))}{\mathbf{E}_1^{\mathbf{P}}[v'(\eta_2-A_2)]^2} \\
& \qquad \cdot\big[\mathbf{P}^c(uu)\cdot v'(\eta_2^{uu}-\alpha\cdot\overline{A}_2(uu))\cdot[\eta_2^{uu}-\eta_2^{uu}] \\
& \qquad +\mathbf{P}^c(um)\cdot v'(\eta_2^{um}-\alpha\cdot\overline{A}_2(um))\cdot[\eta_2^{uu}-\eta_2^{um}] \\
& \qquad +\mathbf{P}^c(ud)\cdot v'(\eta_2^{ud})\cdot\big[\eta_2^{uu}-\eta_2^{ud}\big]\big] \\
& \qquad > 0.
\end{aligned}
$$

Step 2: $(b)-(d)$ yields,

$$\begin{aligned}
&\frac{\mathbf{P}^c(um)\cdot v''(\eta_2^{um}-\alpha\cdot\overline{A}_2(um))\cdot\eta_2^{um}\cdot(-\overline{A}_2(um))\cdot\mathbf{E}_1^{\mathbf{P}}[v'(\eta_2-A_2)]}{\mathbf{E}_1^{\mathbf{P}}[v'(\eta_2-A_2)]^2}\\
&-\frac{\mathbf{P}^c(um)\cdot v''(\eta_2^{um}-\alpha\cdot\overline{A}_2(um))\cdot(-\overline{A}_2(um))\cdot\mathbf{E}_1^{\mathbf{P}}[v'(\eta_2-A_2)\cdot\eta_2]}{\mathbf{E}_1^{\mathbf{P}}[v'(\eta_2-A_2)]^2}\\
&=\frac{\mathbf{P}^c(um)\cdot v''(\eta_2^{um}-\alpha\cdot\overline{A}_2(um))\cdot(-\overline{A}_2(um))}{\mathbf{E}_1^{\mathbf{P}}[v'(\eta_2-A_2)]^2}\\
&\cdot\left[\eta_2^{um}\cdot\mathbf{E}_1^{\mathbf{P}}[v'(\eta_2-A_2)]-\mathbf{E}_1^{\mathbf{P}}[v'(\eta_2-A_2)\cdot\eta_2]\right]\\
&=\frac{\mathbf{P}^c(um)\cdot v''(\eta_2^{um}-\alpha\cdot\overline{A}_2(um))\cdot(-\overline{A}_2(um))}{\mathbf{E}_1^{\mathbf{P}}[v'(\eta_2-A_2)]^2}\\
&\cdot\Big[\mathbf{P}^c(uu)\cdot v'(\eta_2^{uu}-\alpha\cdot\overline{A}_2(uu))\cdot[\eta_2^{um}-\eta_2^{uu}]\\
&+\mathbf{P}^c(um)\cdot v'(\eta_2^{um}-\alpha\cdot\overline{A}_2(um))\cdot[\eta_2^{um}-\eta_2^{um}]\\
&+\mathbf{P}^c(ud)\cdot v'(\eta_2^{ud})\cdot\left[\eta_2^{um}-\eta_2^{ud}\right]\Big]\\
&\lesseqgtr 0
\end{aligned}$$

Step 3: Thus, we have,

$$\frac{\partial S_1^1(u)}{\partial\alpha}\lesseqgtr 0,$$

as asserted. ■

7.8.2 Proof of proposition 71

From the proof of proposition 68, we can directly conclude that here,

$$\frac{\partial S_1^1(u)}{\partial\alpha}\lesseqgtr 0,$$

as well. Since the present setting includes the case $\rho=0$, it remains to show that,

$$\frac{\partial S_1^1(d)}{\partial\alpha}\lesseqgtr 0.$$

Let $A_2\equiv\alpha\cdot\overline{A}_2$ where $\overline{A}_2$ is as defined in (7.29). $S_1^1(d)$ is according to Theorem 64 given as,

$$S_1^1(d)=\frac{\mathbf{E}_1^{\mathbf{P}}[v'(\eta_2-\alpha\cdot\overline{A}_2)\cdot\eta_2]}{\mathbf{E}_1^{\mathbf{P}}[v'(\eta_2-\alpha\cdot\overline{A}_2)]},$$

where,

$$\begin{aligned}\mathbf{E}_1^{\mathbf{P}}[v'(\eta_2 - \alpha \cdot \overline{A}_2) \cdot \eta_2] \;=\; & \frac{1}{\mathbf{P}(du) + \mathbf{P}(dm) + \mathbf{P}(dd)} \\ & \cdot \big[\mathbf{P}(du) \cdot v'(\eta_2^{du}) \cdot \eta_2^{du} \\ & + \mathbf{P}(dm) \cdot v'(\eta_2^{dm} - \alpha \cdot \overline{A}_2(dm)) \cdot \eta_2^{dm} \\ & + \mathbf{P}(dd) \cdot v'(\eta_2^{dd} - \alpha \cdot \overline{A}_2(dd)) \cdot \eta_2^{dd}\big],\end{aligned}$$

and,

$$\begin{aligned}\mathbf{E}_1^{\mathbf{P}}[v'(\eta_2 - \alpha \cdot \overline{A}_2) \cdot \eta_2] \;=\; & \frac{1}{\mathbf{P}(du) + \mathbf{P}(dm) + \mathbf{P}(dd)} \\ & \cdot \big[\mathbf{P}(du) \cdot v'(\eta_2^{du}) \\ & + \mathbf{P}(dm) \cdot v'(\eta_2^{dm} - \alpha \cdot \overline{A}_2(dm)) \\ & + \mathbf{P}(dd) \cdot v'(\eta_2^{dd} - \alpha \cdot \overline{A}_2(dd))\big].\end{aligned}$$

To simplify notation, define the single conditional probabilities by,

$$\forall \omega \in \{du, dm, dd\} : \mathbf{P}^c(\omega) \equiv \frac{\mathbf{P}(\omega)}{\mathbf{P}(du) + \mathbf{P}(dm) + \mathbf{P}(dd)}.$$

One obtains,

$$\begin{aligned}& \frac{\partial S_1^1(d)}{\partial \alpha} \\ = \; & \underbrace{\frac{\mathbf{P}^c(dm) \cdot v''(\eta_2^{dm} - \alpha \cdot \overline{A}_2(dm)) \cdot \eta_2^{dm} \cdot (-\overline{A}_2(dm)) \cdot \mathbf{E}_1^{\mathbf{P}}[v'(\eta_2 - A_2)]}{\mathbf{E}_1^{\mathbf{P}}[v'(\eta_2 - A_2)]^2}}_{(a)} \\ & + \underbrace{\frac{\mathbf{P}^c(dd) \cdot v''(\eta_2^{dd} - \alpha \cdot \overline{A}_2(dd)) \cdot \eta_2^{dd} \cdot (-\overline{A}_2(dd)) \cdot \mathbf{E}_1^{\mathbf{P}}[v'(\eta_2 - A_2)]}{\mathbf{E}_1^{\mathbf{P}}[v'(\eta_2 - A_2)]^2}}_{(b)} \\ & - \underbrace{\frac{\mathbf{P}^c(dm) \cdot v''(\eta_2^{dm} - \alpha \cdot \overline{A}_2(dm)) \cdot (-\overline{A}_2(dm)) \cdot \mathbf{E}_1^{\mathbf{P}}[v'(\eta_2 - A_2) \cdot \eta_2]}{\mathbf{E}_1^{\mathbf{P}}[v'(\eta_2 - A_2)]^2}}_{(c)} \\ & - \underbrace{\frac{\mathbf{P}^c(dd) \cdot v''(\eta_2^{dd} - \alpha \cdot \overline{A}_2(dd)) \cdot (-\overline{A}_2(dd)) \cdot \mathbf{E}_1^{\mathbf{P}}[v'(\eta_2 - A_2) \cdot \eta_2]}{\mathbf{E}_1^{\mathbf{P}}[v'(\eta_2 - A_2)]^2}}_{(d)}.\end{aligned}$$

The strategy to prove the remaining part of the proposition is very similar to the strategy pursued in the proof of proposition 68. We demonstrate in step 1 that $(b)-(d)<0$ and in step 2 that $(a)-(c) \lesseqgtr 0$. In the end, this yields in step 3 the assertion that $(a)+(b)-(c)-(d) \lesseqgtr 0$, or equivalently, $\frac{\partial S_1^1(d)}{\partial \alpha} \lesseqgtr 0$.

Step 1: $(b)-(d)$ yields,

$$
\begin{aligned}
&\frac{\mathbf{P}^c(dd)\cdot v''(\eta_2^{dd}-\alpha\cdot\overline{A}_2(dd))\cdot\eta_2^{dd}\cdot(-\overline{A}_2(dd))\cdot\mathbf{E}_1^{\mathbf{P}}[v'(\eta_2-A_2)]}{\mathbf{E}_1^{\mathbf{P}}[v'(\eta_2-A_2)]^2} \\
-&\frac{\mathbf{P}^c(dd)\cdot v''(\eta_2^{dd}-\alpha\cdot\overline{A}_2(dd))\cdot(-\overline{A}_2(dd))\cdot\mathbf{E}_1^{\mathbf{P}}[v'(\eta_2-A_2)\cdot\eta_2]}{\mathbf{E}_1^{\mathbf{P}}[v'(\eta_2-A_2)]^2} \\
&= \frac{\mathbf{P}^c(dd)\cdot v''(\eta_2^{dd}-\alpha\cdot\overline{A}_2(dd))\cdot(-\overline{A}_2(dd))}{\mathbf{E}_1^{\mathbf{P}}[v'(\eta_2-A_2)]^2} \\
&\cdot\left[\eta_2^{dd}\cdot\mathbf{E}_1^{\mathbf{P}}[v'(\eta_2-A_2)]-\mathbf{E}_1^{\mathbf{P}}[v'(\eta_2-A_2)\cdot\eta_2]\right] \\
&= \frac{\mathbf{P}^c(dd)\cdot v''(\eta_2^{dd}-\alpha\cdot\overline{A}_2(dd))\cdot(-\overline{A}_2(dd))}{\mathbf{E}_1^{\mathbf{P}}[v'(\eta_2-A_2)]^2} \\
&\cdot\left[\mathbf{P}^c(du)\cdot v'(\eta_2^{du})\cdot\left[\eta_2^{dd}-\eta_2^{du}\right]\right. \\
&+\mathbf{P}^c(dm)\cdot v'(\eta_2^{dm}-\alpha\cdot\overline{A}_2(dm))\cdot\left[\eta_2^{dd}-\eta_2^{dm}\right] \\
&\left.+\mathbf{P}^c(dd)\cdot v'(\eta_2^{dd}-\alpha\cdot\overline{A}_2(dd))\cdot\left[\eta_2^{dd}-\eta_2^{dd}\right]\right] \\
&< 0.
\end{aligned}
$$

Step 2: $(a)-(c)$ yields,

$$
\begin{aligned}
&\frac{\mathbf{P}^c(dm)\cdot v''(\eta_2^{dm}-\alpha\cdot\overline{A}_2(dm))\cdot\eta_2^{dm}\cdot(-\overline{A}_2(dm))\cdot\mathbf{E}_1^{\mathbf{P}}[v'(\eta_2-A_2)]}{\mathbf{E}_1^{\mathbf{P}}[v'(\eta_2-A_2)]^2} \\
-&\frac{\mathbf{P}^c(dm)\cdot v''(\eta_2^{dm}-\alpha\cdot\overline{A}_2(dm))\cdot(-\overline{A}_2(dm))\cdot\mathbf{E}_1^{\mathbf{P}}[v'(\eta_2-A_2)\cdot\eta_2]}{\mathbf{E}_1^{\mathbf{P}}[v'(\eta_2-A_2)]^2} \\
&= \frac{\mathbf{P}^c(dm)\cdot v''(\eta_2^{dm}-\alpha\cdot\overline{A}_2(dm))\cdot(-\overline{A}_2(dm))}{\mathbf{E}_1^{\mathbf{P}}[v'(\eta_2-A_2)]^2} \\
&\cdot\left[\eta_2^{dm}\cdot\mathbf{E}_1^{\mathbf{P}}[v'(\eta_2-A_2)]-\mathbf{E}_1^{\mathbf{P}}[v'(\eta_2-A_2)\cdot\eta_2]\right] \\
&= \frac{\mathbf{P}^c(dm)\cdot v''(\eta_2^{dm}-\alpha\cdot\overline{A}_2(dm))\cdot(-\overline{A}_2(dm))}{\mathbf{E}_1^{\mathbf{P}}[v'(\eta_2-A_2)]^2} \\
&\cdot\left[\mathbf{P}^c(du)\cdot v'(\eta_2^{du})\cdot\left[\eta_2^{dm}-\eta_2^{du}\right]\right. \\
&+\mathbf{P}^c(dm)\cdot v'(\eta_2^{dm}-\alpha\cdot\overline{A}_2(dm))\cdot\left[\eta_2^{dm}-\eta_2^{dm}\right] \\
&\left.+\mathbf{P}^c(dd)\cdot v'(\eta_2^{dd}-\alpha\cdot\overline{A}_2(dd))\cdot\left[\eta_2^{dm}-\eta_2^{dd}\right]\right] \\
&\lesseqgtr 0.
\end{aligned}
$$

Step 3: Hence, we have,

$$\frac{\partial S_1^1(d)}{\partial \alpha} \lesseqqgtr 0,$$

as desired. ■

Concluding Remarks

This thesis was comprised of three parts. The introductory part of the thesis, part I, was intended to introduce the reader to dynamic hedging and several aspects related to it. In chapter 1, we discussed the main principles in which dynamic hedging programs are generally based on. We saw that *replication* and *arbitrage* are powerful concepts in deriving fair prices for contingent claims like options, and in deriving appropriate dynamic hedging strategies. We delineated the assumptions under which replication and arbitrage work perfectly. Typically, dynamic hedging strategies produce positive feedback. Such a trading behavior is incompatible with the standard rationality hypothesis that individual behavior is representable as the maximization of a suitably chosen utility function. Therefore, we introduced in chapter 1 the *noise trader approach* which is capable of incorporating agents who act 'irrational'. We argued that this approach permits the explanation of several puzzling observations that are incompatible with the standard paradigm of efficient markets. Chapter 1 then proceeded by examining the relationship between dynamic hedging and *market liquidity*. A simple example demonstrated that the standard replication-arbitrage approach breaks down in imperfectly liquid markets. Empirical results and two case studies supported our case for the importance of market liquidity for dynamic hedging.

Presently, the study of dynamic hedging in imperfectly liquid markets represents an active field of research in financial economics. Chapter 2 surveyed articles that fall in this strand of literature. In the spirit of the noise trader approach, such studies generally consider equilibrium models where rational agents interact with irrational agents like hedgers. The studies mainly differ in one dimension, namely, whether information is complete or incomplete. Regardless of the actual information structure almost all studies produce similar results. For example, all but one author observe that dynamic hedging increases volatility. Since market volatility is generally considered to be a measure for the stability of financial markets, this is a really striking result. We concluded chapter 2, and thereby part I of the thesis, by outlining some possible applications of the models proposed in the surveyed articles.

Part II of the thesis laid the theoretical foundations for our formal analysis. Chapter 3 presented a brief review of several aspects related to *uncertainty* in financial economics. It introduced a formal model capable of capturing the basic notions of uncertainty in a financial market and sketched an approach to decision making under uncertainty. Chapter 4 built on the analysis in chapter 3. Chapter 4's focus was on the *martingale approach* to finance. It presented a general model framework and derived central results with respect to the absence of arbitrage, the existence of an equivalent martingale measure and the pricing of contingent claims. Examples illustrated the application of some of these results.

In part III of the thesis, we applied the tools and methods of part II to three different economic settings. Chapter 5 set the stage in that it analyzed dynamic hedging in a 'perfect' world. The main result of this chapter was that dynamic hedging of contingent claims with convex payoffs produces *positive feedback*. The striking point about this result is that almost all contingent claims engineered in the real marketplace have convex payoffs. The positive feedback result was illustrated by examples utilizing the Black / Scholes pricing formula.

In chapter 6, we considered a *general equilibrium* model to investigate the impact of dynamic hedging on financial markets. Following the noise trader approach, the model economy was populated by hedgers following dynamic hedge programs and non-hedgers maximizing their expected utility. It turned out that a unique general equilibrium existed and that the market model was complete in equilibrium under a common knowledge assumption. For certain call and put options, we showed in chapter 6 that the implementation of their corresponding hedging strategies inevitably increases volatility. However, this result heavily hinges on the particular class of contingent claim considered. Examples demonstrated that positive feedback trading by hedgers may both *increase* and *decrease* the volatility of the underlying stock. On a similar note, another example revealed that negative feedback trading may increase volatility. These findings are in sharp contrast to findings reported in similar studies. We provided an explanation for our observations that was mainly based on arguments concerning the market liquidity. In these particular findings we see one of our main contributions to the literature.

Chapter 7 generalized the market model of chapter 6 to incomplete markets. The study of dynamic hedging in a general equilibrium framework with inherent market incompleteness is new. It therefore represents one of our main contributions as well. We connected the field of incomplete markets research with the field of research concerned with general equilibrium effects of dynamic hedging. Generally speaking, perfect hedges are no longer feasible in incomplete markets. In light of this, we required the hedgers to super-replicate so that they at least achieve a complete hedge. Contrary to a complete markets setting, super-replication in incomplete markets may generate price process-dependent optimal strategies. Yet we showed that the strategies for super-replicating those call and put options, which we were mainly interested in, are price-independent. Assuming that hedgers only hedge these call and put options, a unique general equilibrium existed in the market model of chapter 7. Our analysis revealed that super-replication smooths the payoff that the hedgers actually achieve. Due to this smoothing effect, it was no longer clear what impact dynamic hedging has. Whereas we found that dynamic hedging of calls and puts may produce clear cut price and volatility effects in the complete markets framework of chapter 6, this result broke down in the incomplete markets framework of chapter 7. By using numerical simulations, we compared the quantitative effects dynamic hedging has in the frameworks of chapters 6 and 7, respectively.

The main conclusion that we can draw from our results is that dynamic hedging - typically posing positive feedback on financial markets - does not necessarily destabilize these markets. Taking volatility as a measure for market stability, we saw that positive feedback hedging may both stabilize or destabilize markets. These results are in sharp contrast to those reported in a number of similar studies. In other words, the results of these studies seem to be artifacts of the respective parameter specifications. Even though we did not obtain clear cut results regarding the direction in which dynamic hedging influences financial markets, we saw in the examples that market liquidity is an important determinant of the actual impact of dynamic hedging on financial markets.

If we would have to pick the crucial assumption that primarily drove our results, we would pick the assumption of complete and symmetric information among agents. From an economic point of view, it seems risky to presume, for instance, that every agent is aware of the extent to which dynamic hedging takes place and at which terms it does so. Certainly, it would be interesting to analyze the implications of relaxing this assumption in our model framework. This could be done by modelling the parameter α - the market weight of hedgers - as a random variable or even as an uncertain quantity.

Another critical assumption we made once in a while concerned the payoff structure of the contingent claims under consideration. Particularly in chapter 7, the results hinge heavily on the circumstance that the strategy of the hedgers is price-independent. To examine more general classes would be an interesting direction for future research as well.

The assumption made in chapters 6 and 7 that only three dates are relevant represents a drawback in technical terms rather than in economic terms. For example, our notion of volatility presented in chapters 6 and 7 clearly depends on the particular model structure. Yet the extension of our general equilibrium models to more than three dates is straightforward and would only raise the question of how volatility should be measured in such an environment. We are quite confident that our results carry over to such a generalized setting and do not see a problem in finding an appropriate measure for the volatility.

Of course, there are still a lot of other aspects related to dynamic hedging that deserve more attention in future research. Our hope is, however, that this thesis helped to clarify - in economic terms - some of the more important aspects of dynamic hedging.

We started our exposition with a discussion of the BLACK / SCHOLES / MERTON approach to option pricing and hedging. We also reported that two of these outstanding researchers, ROBERT MERTON and MYRON SCHOLES, had to witness how their investment company LTCM - mainly investing according to their own approach - nearly collapsed. Therefore, it seems quite natural to conclude this thesis by quoting one of the people who invented the approach that revolutionized the financial services industry and that considerably influenced the thinking of researchers in finance. ROBERT MERTON'S statement below, made on the occasion of the NOBEL prize ceremony and therewith before the disastrous events in the second half of 1998, expresses his concerns about the applicability of financial models to the real world. The models and results presented in this thesis should be judged in the light of his concise warning.

> "*The mathematics of financial models can be applied precisely, but the models are not at all precise in their application to the complex real world. Their accuracy as a useful approximation to that world varies significantly across time and place. The models should be applied in practice only tentatively, with careful assessment of their limitations in each application.*" MERTON (1998, 343).

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